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Spontaneous breaking of SO(3) to finite family symmetries with supersymmetry — an A_4 model

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Abstract: We discuss the breaking of SO(3) down to finite family symmetries such as A_4, S_4 and A_5 using supersymmetric potentials for the first time. We analyse in detail the case of supersymmetric A_4 and its finite subgroups Z_3 and Z_2. We then propose a supersymmetric A_4 model of leptons along these lines, originating from $\mathrm{SO}(3) \times \mathrm{U}(1)$, which leads to a phenomenologically acceptable pattern of lepton mixing and masses once subleading corrections are taken into account. We also discuss the phenomenological consequences of having a gauged SO(3), leading to massive gauge bosons, and show that all domain wall problems are resolved in this model.

Keywords: Gauge Symmetry, Discrete Symmetries, Neutrino Physics, Supersymmetric Effective Theories

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Contents

1 Introduction

The discovery of neutrino mass and lepton mixing [1] not only represents the first laboratory particle physics beyond the Standard Model (BSM) but also raises additional flavour puzzles such as why the neutrino masses are so small, and why lepton mixing is so large [2]. Early family symmetry models focussed on continuous non-Abelian gauge theories such as SU(3) [3, 4][1] or SO(3) [6–8]. Subsequently, non-Abelian discrete symmetries such as A_4 were introduced, for example to understand the theoretical origin of the observed pattern of (approximate) tri-bimaximal lepton mixing [9–11]. When supersymmetry (SUSY) is included, the problem of vacuum alignment which is crucial to the success of such theories, can be more readily addressed using the flat directions of the potential [12–15]. However, current data involves a non-zero reactor angle and a solar angle which deviate from their tri-bimaximal values [16]. Since, in general, non-Abelian discrete symmetries do not imply either a zero reactor angle or exact tri-bimaximal lepton mixing, these symmetries are still widely used in current model building [17–19].

Although the motivation for non-Abelian discrete symmetries remains strong, there are a few question marks surrounding the use of such symmetries in physics. The first and most obvious question is from where do such symmetries originate? In the Standard Model (SM) we are familiar with the idea of gauge theories being fundamental and robust symmetries of nature, but discrete symmetries seem only relevant to charge conjugation (C), parity (P) and time-reversal invariance (T) symmetry [20]. In supersymmetric (SUSY) models, Abelian discrete symmetries are commonly used to ensure proton stability [21]. It is possible that the non-Abelian discrete symmetries could arise from some high energy theory such as string theory [22], perhaps as a subgroup of the modular group [13, 23–28] and/or from the orbifolding of extra dimensions [29–32]. However, even if such symmetries do arise from string theory, and survive quantum and gravitational corrections [33], when they are spontaneously broken they would imply that distinct degenerate vacua exist separated by an energy barrier, leading to a network of cosmological domain walls which would be in conflict with standard cosmology, and appear to "over-close the Universe" [34–36].

The problem of domain walls with non-Abelian discrete symmetries such as A_4 was discussed in [37, 38] where three possible solutions were discussed:

1. to suppose that the A_4 discrete symmetry is anomalous, and hence it is only a symmetry of the classical action and not a full symmetry of the theory, being broken by quantum corrections. For example this could be due to extending the discrete symmetry to the quark sector such that the symmetry is broken at the quantum level due to the QCD anomaly [39]. However, it is not enough to completely solve the problem since this anomaly cannot remove all the vacuum degeneracy [40];

2. to include explicit A_4 breaking terms in the Lagrangian, possibly in the form of Planck scale suppressed higher order operators, arising from gravitational effects;

3. to suppose that, in the thermal history of the Universe, the A_4 breaking phase transition happens during inflation which effectively dilutes the domain walls, and that the A_4 is never restored after reheating following inflation.

[1] SU(3) has recently been considered in extra dimensions [5].

An alternative solution to the domain wall problem, which we pursue here, is to suppose that the non-Abelian discrete symmetry arises as a low energy remnant symmetry after the spontaneous breaking of some non-Abelian continuous gauge theory. This could take place either within the framework of string theory [41], or, as in the present paper, in the framework of quantum field theory (QFT). For example it has been shown how SO(3) can be spontaneously broken to various non-Abelian discrete symmetries [42, 43]. In order to achieve this, a scalar potential was constructed such leading to the vaccuum expectation value (VEV) which breaks the continuous gauge symmetry to the discrete symmetry. The key requirement for having a remnant non-Abelian discrete symmetry seems to be that the scalar field which breaks the gauge symmetry is in some large irreducible representation (irrep) of the continuous gauge group.

The above approach [42, 43] has been applied to flavour models based on non-Abelian continuous gauge symmetries. For example, following [42, 43], the authors in [44] have considered the breaking of gauged SO(3) $\to A_4$ by introducing $\underline{7}$-plet of SO(3) with the further breaking of A_4 realising tri-bimaximal mixing in a non-SUSY flavour model. However, a fine-tuning of around 10^{-2} among parameters had to be considered in order to get the correct hierarchy between μ and τ masses. The problem of how to achieve tri-bimaximal mixing at leading order from non-Abelian continuous flavour symmetries has also been discussed by other authors [45, 46] but the problem of determining the required flavon VEVs remains unclear. One idea is to require the electroweak doublets and right-handed fermions to separately transforming under different continuous flavour symmetries, and realise maximal atmospheric mixing from the minimisation of the potential [47, 48]. Extended discussions including the breaking of SU(2) and SU(3) to non-Abelian discrete symmetries have been discussed in [49–54] and the phenomenological implications of the breaking of SU(3) flavour symmetry in flavour models has been discussed in [55, 56].

The above literature has been concerned with breaking a continuous gauge theory to a non-Abelian discrete symmetry *without SUSY*. To date, the problem of how to achieve such a breaking *in a SUSY framework* has not been addressed, even though there are many SUSY flavour models in the literature [17–19]. As stated earlier, the main advantage of such SUSY models is the possibility to achieve vacuum alignment using flat directions of the potential, which enables some technical simplifications and enhances the theoretical stability of the alignment [12]. There is also a strong motivation for considering such breaking in a SUSY framework, in order to make contact with *SUSY flavour models* [17–19]. In addition, the usual motivations for embedding the non-Abelian discrete symmetry into a gauge theory also apply in the SUSY context as well, namely:

- To provide a natural explanation of the origin of non-Abelian discrete flavour symmetries in SUSY flavour models.

- To avoid the domain wall problem of SUSY flavour models, since the non-Abelian discrete flavour symmetry is just an approximate effective residual symmetry arising from the breaking of the continuous symmetry. When the approximate discrete symmetry is broken it does not lead to domain walls.

JHEP11(2018)173

Finally, if the continuous symmetry is gauged, there is the phenomenological motivation that:

- The breaking of *gauged flavour symmetries* to finite non-Abelian flavour symmetries implies new massive gauge bosons in the spectrum, with possibly observable phenomenological signatures. For instance, SUSY SO(3) $\to A_4$ will lead to three degenerate gauge bosons plus their superpartners.

In the present paper, motivated by the above considerations, we discuss the breaking of a continuous *SUSY* gauge theory to a non-Abelian discrete symmetry using a potential which *preserves SUSY*. As stated above, this is the first time that such a symmetry breaking has been discussed in the literature, and the formalism developed here may be applied to the numerous SUSY flavour models in the literature [17–19]. For example, we discuss the breaking of SO(3) down to finite family symmetries such as A_4, S_4 and A_5 using supersymmetric potentials for the first time. In particular, we focus in detail on the breaking of SUSY SO(3) to A_4, with SUSY preserved by the symmetry breaking. We further show how the A_4 may be subsequently broken to smaller residual symmetries Z_3 and Z_2, still preserving SUSY, which may be used to govern the mixing patterns in the charged lepton and neutrino sectors, leading to a predictive framework. We then present an explicit SUSY SO(3) $\times$ U(1) model of leptons which uses this symmetry breaking pattern and show that it leads to a phenomenologically acceptable pattern of lepton mixing and masses. Finally we discuss the phenomenological consequences of having a gauged SO(3), leading to massive gauge bosons, and show that all domain wall problems are resolved in such models.

The layout of the remainder of the paper is then as follows. In section 2 we discuss the spontaneous breaking of SO(3) to finite non-Abelian symmetries such as A_4, S_4 and A_5 with supersymmetry. In section 3 we discuss the further breaking of A_4 to residual Z_3 and Z_2 symmetries, showing how it may be achieved from a supersymmetric SO(3) potential. In section 4 we construct in detail a supersymmetric A_4 model along these lines, originating from SO(3) $\times$ U(1), and show that it leads to a phenomenologically acceptable pattern of lepton mixing and masses, once subleading corrections are taken into account. Within this model, we also discuss the phenomenological consequences of having a gauged SO(3), leading to massive gauge bosons, and show that all domain wall problems are resolved. Section 5 concludes the paper. The paper has three appendices. In appendix A we list the Clebsch-Gordan coefficients of SO(3) which are used in the paper. In appendix B we display explicitly the solutions of the superpotential minimisation. In appendix C we show the deviation from the Z_2-invariant vacuum.

2 Spontaneous breaking of SO(3) to finite non-Abelian symmetries A_4, S_4 and A_5 with supersymmetry

The key point to break SO(3) to non-Abelian discrete symmetries is introducing a high irrep of SO(3) and require it gain a non-trivial VEV. In this section, after a brief review of SO(3), we discuss how to break SO(3) to A_4 by introducing a $\underline{7}$-plet, and then generalise our discussion to SO(3) $\to S_4$ and A_5.

2.1 The SO(3) group

The rotation group SO(3) is one of the most widely used Lie groups in physics and mathematics. It is generated by three generators τ^1, τ^2 and τ^3. Each element can be expressed by

$$g_{\{\alpha^a\}} = \exp\left(\sum_{a=1,2,3} \alpha^a \tau^a\right) = \mathbb{1} + \sum_{a=1,2,3} \alpha^a \tau^a + \frac{1}{2}\left(\sum_{a=1,2,3} \alpha^a \tau^a\right)^2 + \cdots . \tag{2.1}$$

In the fundamental three dimensional (3d) space, the generators are represented as

$$\tau^1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.2}$$

Each irrep of SO(3) has $2p+1$ dimensions and we denote it as a $\underline{2p{+}1}$-plet. Each $\underline{2p{+}1}$-plet can be represented as a rank-p tensor $T_{i_1 i_2 \ldots i_p}$ in the 3d space. This tensor is symmetric and traceless,

$$\phi_{\ldots i_a \ldots i_b \ldots} = \phi_{\ldots i_b \ldots i_a \ldots}\,, \qquad \sum_{i_a = i_b = 1}^{3} \phi_{\ldots i_a \ldots i_b \ldots} = 0\,, \tag{2.3}$$

for any $a, b \leqslant p$. It transforms under SO(3) as

$$\phi_{i_1 i_2 \ldots i_p} \to O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_p j_p} \phi_{j_1 j_2 \ldots j_p}\,, \tag{2.4}$$

where O is transformation matrix corresponding to the element $g_{\{\alpha^a\}}$ in the 3d space, and it is always a 3×3 real orthogonal matrix. Here and in the following, doubly repeated indices are summed.

Products of two irreps can be reduced as $\underline{2p{+}1} \times \underline{2q{+}1} = \underline{2|p{-}q|{+}1} + \underline{2|p{-}q|{+}3} + \cdots + \underline{2(p{+}q){+}1}$ and the Clebsch-Gordan coefficients are given in appendix A.

2.2 SO(3) $\to$ non-Abelian discrete symmetries

SO(3) can be spontaneously broken to other non-Abelian discrete symmetries by introducing different high irreps. Ref. [43] gives an incomplete list of subgroups which could be obtained after the relevant irrep get a VEV. For instance, some of those subgroup obtained by irreps up to $\underline{13}$ are shown in table 1. The minimal irrep for SO(3) $\to S_4$ is a $\underline{9}$-plet, while that for SO(3) $\to A_5$ is a $\underline{13}$-plet. Applying a $\underline{9}$-plet flavon ρ and a $\underline{13}$-plet flavon ψ, respectively, we will realise these breakings in a SUSY framework in the following.

2.2.1 SO(3) $\to A_4$

The simplest irrep to break SO(3) $\to A_4$ is using a $\underline{7}$-plet [42, 43]. In this work, we introduce a $\underline{7}$-plet flavon ξ to achieve this goal. In the 3d flavour space, it is represented as a rank-3 tensor ξ_{ijk}, which satisfies the requirements in eq. (2.3), i.e.,

$$\xi_{ijk} = \xi_{jki} = \xi_{kij} = \xi_{ikj} = \xi_{jik} = \xi_{kji}\,, \quad \xi_{iik} = 0\,. \tag{2.5}$$

JHEP11(2018)173

irrep	$\underline{1}$	$\underline{3}$	$\underline{5}$	$\underline{7}$	$\underline{9}$	$\underline{11}$	$\underline{13}$
subgroups	SO(3)	SO(2)	$Z_2 \times Z_2$	$\mathbf{1}$	S_4		$\mathbf{1}$
		SO(3)	SO(2)	A_4			A_4
			SO(3)	Z_3			S_4
				D_4			A_5
				SO(2)			
				SO(3)			

Table 1. The not systematical stabiliser subgroups in the low-dimensional irreducible representations of the group SO(3) [43].

Constrained by eq. (2.5), there are 7 free components of ξ, which can be chosen as

$$\xi_{111},\, \xi_{112},\, \xi_{113},\, \xi_{123},\, \xi_{133},\, \xi_{233},\, \xi_{333}\,. \tag{2.6}$$

For the A_4 symmetry, we work in the Ma-Rajasekaran (MR) basis, where the generators s and t in the 3d irreducible representation are given by

$$g_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_t = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.7}$$

The A_4-invariant VEV, satisfying

$$\begin{aligned} (g_s)_{ii'}(g_s)_{jj'}(g_s)_{kk'}\langle \xi_{i'j'k'}\rangle &= \langle \xi_{ijk}\rangle\,, \\ (g_t)_{ii'}(g_t)_{jj'}(g_t)_{kk'}\langle \xi_{i'j'k'}\rangle &= \langle \xi_{ijk}\rangle\,, \end{aligned} \tag{2.8}$$

is given by

$$\langle \xi_{123}\rangle \equiv \frac{v_\xi}{\sqrt{6}}\,, \quad \langle \xi_{111}\rangle = \langle \xi_{112}\rangle = \langle \xi_{113}\rangle = \langle \xi_{133}\rangle = \langle \xi_{233}\rangle = \langle \xi_{333}\rangle = 0\,. \tag{2.9}$$

The VEV of ξ is geometrically shown in figure 1.

The discussion of SO(3) $\to A_4$ has been given in refs. [42–44]. The main idea is constructing flavon potential and clarifying the A_4-invariant one in eq. (2.9) to be the minimum of the potential, where v_ξ is determined by the minimisation. This idea cannot be directly applied to supersymmetric flavour models. In the later case, the flavon potential is directly related to the flavon superpotential

$$V_f = \sum_i \left| \frac{\partial w_f}{\partial \phi_i} \right|^2 + \cdots\,, \tag{2.10}$$

where ϕ_i represent any scalars in the theory, and the dots are negligible soft breaking terms and D-terms for the fields charged under the gauge group. This potential is more constrained than the non-supersymmetric version. If the minimisation of the superpotential $\partial w_f/\partial \phi_i = 0$ has a solution, the minimisation of the potential $\partial V_f/\partial \phi_i = 0$ is identical

JHEP11(2018)173

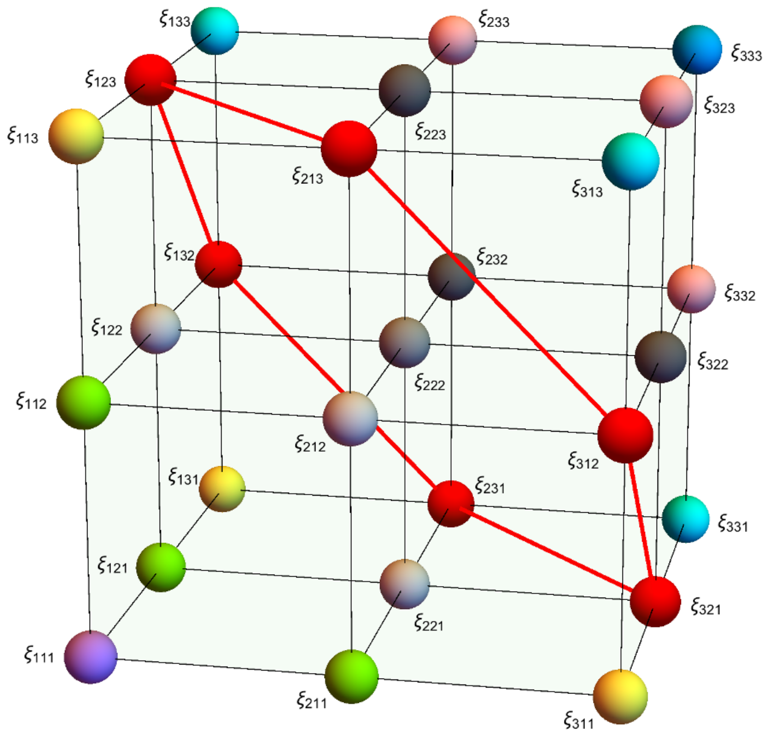

Figure 1. A geometrical description of the $\underline{7}$-plet ξ_{ijk} as a tank-3 tensor with $i, j, k = 1, 2, 3$. Points in the same colour represent the identical components, e.g., $\xi_{112} = \xi_{121} = \xi_{211}$ all in green, etc. As a traceless tensor, points in grey are dependent upon the rest, e.g., $\xi_{122} = \xi_{212} = \xi_{221} = -\xi_{111} - \xi_{133}$. These properties leave only 7 independent components, showing in 7 different colours. For the A_4-invariant VEV, only those in red, $\xi_{123} = \xi_{132} = \xi_{231} = \xi_{213} = \xi_{312} = \xi_{321}$, take non-zero values.

to the minimisation of the superpotential. Since most flavour models have been built in SUSY, it is necessary to consider if SO(3) $\to A_4$ can be achieved in SUSY.

In order to break SO(3) to A_4, we introduce two driving fields $\xi^d_{\underline{1}} \sim \underline{1}$, $\xi^d_{\underline{5}} \sim \underline{5}$ and consider the following superpotential terms

$$w_\xi = \xi^d_{\underline{1}} \left(c_1(\xi\xi)_{\underline{1}} - \mu^2_\xi\right) + c_2\left(\xi^d_{\underline{5}}(\xi\xi)_{\underline{5}}\right)_{\underline{1}}, \tag{2.11}$$

where c_1 and c_2 are complex dimensionless coefficients. As required [13], the driving fields do not gain non-zero VEVs, realised by imposing $\mathrm{U}(1)_R$ charges. Minimisation of the potential is identical to the minimisation of the flavon superpotential respecting to the driving fields as follows,

$$\frac{\partial w_\xi}{\partial \xi^d_{\underline{1}}} = c_1(\xi\xi)_{\underline{1}} - \mu^2_\xi = 0\,, \tag{2.12}$$

$$\frac{\partial w_\xi}{\partial \xi^d_{\underline{5}}} = c_2(\xi\xi)_{\underline{5}} = 0\,. \tag{2.13}$$

The explicit expressions of eqs. (2.12) and (2.13) are listed in appendix B. Taking the A_4-invariant VEV to eqs. (2.12) and (2.13), we see that eq. (2.13) is automatically satisfied

JHEP11(2018)173

and eq. (2.12) leads to $\langle\xi_{123}\rangle = \pm\mu_\xi/\sqrt{6c_1}$. Therefore, the A_4 symmetry is consistent with the vacuum solution obtained from the minimisation of the superpotential.

We need to check *the uniqueness of* A_4 since it is not clear if A_4 is the only symmetry after SO(3) breaking. We assume there is another vacuum solution $\langle\xi\rangle'$, which has an infinitesimal deviation from the A_4-invariant one, $\langle\xi\rangle' = \langle\xi\rangle + \delta\xi$. Eqs. (2.12) and (2.13) must also be satisfied for $\langle\xi\rangle'$. Directly taking them into account, we get the constraints on $\delta\xi$. Straightforwardly, we obtain

$$\delta\xi_{123} = \delta\xi_{111} = \delta\xi_{333} = 0\,, \quad \delta\xi_{112} + \delta\xi_{233} = 0\,, \tag{2.14}$$

leaving only three unconstrained parameters $\delta\xi_{112}$, $\delta\xi_{113}$ and $\delta\xi_{133}$. The unconstrained perturbation parameters $\delta\xi$ can be rotated away if we consider a SO(3) basis transformation, $g_{\{\alpha^a\}}$ in eq. (2.1) with $\alpha^1 = \sqrt{\frac{3c_1}{2}}\delta\xi_{113}/\mu$, $\alpha^2 = \sqrt{\frac{3c_1}{2}}\delta\xi_{112}/\mu$, $\alpha^3 = -\sqrt{\frac{3c_1}{2}}\delta\xi_{133}/\mu$ and the generators τ^i being given in eq. (2.2). Therefore, $\langle\xi\rangle'$ also preserves the A_4 symmetry and the shift from $\langle\xi\rangle$ to $\langle\xi\rangle'$ corresponds to only a basis transformation of SO(3). Such a basis transformation has no physical meaning. We conclude that the minimisation equation of the superpotential, i.e., eqs. (2.12) and (2.13), uniquely breaks SO(3) to A_4.

2.2.2 SO(3) $\to$ S_4

For the S_4 symmetry, the generators in the 3d irreducible space are given by g_s, g_t and

$$g_u = -\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.15}$$

In the 3d flavour space, the $\underline{9}$-plet ρ is represented as a rank-4 tensor ρ_{ijkl}. Constrained by eq. (2.3), there are 9 free components of ρ, which can be chosen as

$$\rho_{1111},\ \rho_{1112},\ \rho_{1113},\ \rho_{1123},\ \rho_{1133},\ \rho_{1233},\ \rho_{1333},\ \rho_{2333},\ \rho_{3333}\,. \tag{2.16}$$

In order to require the VEV $\langle\rho\rangle$ invariant under the S_4 symmetry. The following constraints are required,

$$\begin{aligned}
(g_s)_{ii'}(g_s)_{jj'}(g_s)_{kk'}(g_s)_{ll'}\langle\rho_{i'j'k'l'}\rangle &= \langle\rho_{ijkl}\rangle\,, \\
(g_t)_{ii'}(g_t)_{jj'}(g_t)_{kk'}(g_t)_{ll'}\langle\rho_{i'j'k'l'}\rangle &= \langle\rho_{ijkl}\rangle\,, \\
(g_u)_{ii'}(g_u)_{jj'}(g_u)_{kk'}(g_u)_{ll'}\langle\rho_{i'j'k'l'}\rangle &= \langle\rho_{ijkl}\rangle\,,
\end{aligned} \tag{2.17}$$

which are equivalent to

$$\begin{aligned}
&\langle\rho_{1111}\rangle = \langle\rho_{3333}\rangle = -2\langle\rho_{1133}\rangle\,, \\
&\langle\rho_{1112}\rangle = \langle\rho_{1113}\rangle = \langle\rho_{1123}\rangle = \langle\rho_{1233}\rangle = \langle\rho_{1333}\rangle = \langle\rho_{2333}\rangle = 0\,.
\end{aligned} \tag{2.18}$$

JHEP11(2018)173

Follwing a similar procedure but replacing the $\underline{7}$-plet ξ by a $\underline{9}$-plet ρ, we succeed to break SO(3) to S_4 in SUSY by introducing two driving fields $\rho^d_{\underline{1}} \sim \underline{1}$ and $\rho^d_{\underline{5}} \sim \underline{5}$. The flavon superpotential is constructed as

$$w_\rho = \rho^d_{\underline{1}} \left(\mu_\rho^2 - c_{\rho 1}(\rho\rho)_{\underline{1}}\right) + c_{\rho 2}\rho^d_{\underline{5}}(\rho\rho)_{\underline{5}} \,. \tag{2.19}$$

Minimisation respect to the driving fields gives rise to

$$\frac{\partial w_\rho}{\partial \rho^d_{\underline{1}}} = \mu_\rho^2 - c_{\rho 1}(\rho\rho)_{\underline{1}} = 0 \,, \tag{2.20}$$

$$\frac{\partial w_\rho}{\partial \rho^d_{\underline{5}}} = c_{\rho 2}(\rho\rho)_{\underline{5}} = 0 \,. \tag{2.21}$$

Taking eq. (2.18) to the above equations, we see that eq. (2.21) is automatically satisfied and eq. (2.20) leads to $\langle \rho_{1133}\rangle = \pm \mu_\rho/\sqrt{30 c_{\rho 1}}$.

The uniqueness of SO(3) $\to$ S_4. We vary ρ away from the S_4-invariant VEV, $\rho \to \langle\rho\rangle + \delta\rho$ and require that eqs. (2.20) and (2.21) are still satisfied. Then, we will get the constraints on $\delta\rho$, which are straightforwardly expressed as

$$\delta\rho_{1111} = \delta\rho_{1123} = \delta\rho_{1133} = \delta\rho_{1233} = \delta\rho_{3333} = 0 \,, \quad \delta\rho_{1113} + \delta\rho_{1333} = 0 \,, \tag{2.22}$$

leaving only three unconstrained parameters $\delta\rho_{1112}$, $\delta\rho_{1113}$ and $\delta\rho_{2333}$. The unconstrained perturbation parameters $\delta\rho$ can be rotated away if we consider a SO(3) basis transformation, $g_{\{\alpha^a\}} = \mathbb{1}_{3\times 3} + \alpha^a \tau^a$ with $\alpha^1 = \sqrt{\frac{6c_{\rho 1}}{5}}\delta\rho_{1112}/\mu$, $\alpha^2 = \sqrt{\frac{6c_{\rho 1}}{5}}\delta\rho_{1113}/\mu$, $\alpha^3 = -\sqrt{\frac{6c_{\rho 1}}{5}}\delta\rho_{2333}/\mu$. Therefore, eqs. (2.20) and (2.21) uniquely break SO(3) to S_4.

2.2.3 SO(3) $\to$ A_5

For the A_5 symmetry, the generators in the 3d irreducible space are given by g_s, g_t and

$$g_w = -\frac{1}{2}\begin{pmatrix} -1 & b_2 & b_1 \\ b_2 & b_1 & -1 \\ b_1 & -1 & b_2 \end{pmatrix} \,, \tag{2.23}$$

where $b_1 = \frac{1}{2}(\sqrt{5}-1)$ and $b_2 = \frac{1}{2}(-\sqrt{5}-1)$.

The $\underline{13}$-plet ψ in the 3d flavour space is represented as a rank-6 tensor ψ_{ijklmn}. Constrained by eq. (2.3), there are 13 free components of ψ, which can be chosen to be

$$\psi_{111111},\, \psi_{111112},\, \psi_{111113},\, \psi_{111123},\, \psi_{111133},\, \psi_{111233},\, \psi_{111333},\, \psi_{112333}, \tag{2.24}$$

$$\psi_{113333},\, \psi_{123333},\, \psi_{133333},\, \psi_{233333},\, \psi_{333333} \,. \tag{2.25}$$

In order to require the VEV $\langle\psi\rangle$ invariant under the S_4 symmetry. The following constraints are required,

$$\begin{aligned}
(g_s)_{ii'}(g_s)_{jj'}(g_s)_{kk'}(g_s)_{ll'}(g_s)_{mm'}(g_s)_{nn'}\langle\psi_{i'j'k'l'm'n'}\rangle &= \langle\psi_{ijklmn}\rangle \,, \\
(g_t)_{ii'}(g_t)_{jj'}(g_t)_{kk'}(g_t)_{ll'}(g_t)_{mm'}(g_t)_{nn'}\langle\psi_{i'j'k'l'm'n'}\rangle &= \langle\psi_{ijklmn}\rangle \,, \\
(g_w)_{ii'}(g_w)_{jj'}(g_w)_{kk'}(g_w)_{ll'}(g_w)_{mm'}(g_w)_{nn'}\langle\psi_{i'j'k'l'm'n'}\rangle &= \langle\psi_{ijklmn}\rangle \,.
\end{aligned} \tag{2.26}$$

They are equivalent to

$$\begin{aligned}
&\langle\psi_{111111}\rangle=\langle\psi_{333333}\rangle,\ \langle\psi_{111133}\rangle=\frac{7\sqrt{5}-5}{10}\langle\psi_{111111}\rangle,\ \langle\psi_{113333}\rangle=\frac{-7\sqrt{5}-5}{10}\langle\psi_{111111}\rangle\,,\\
&\langle\psi_{111112}\rangle=\langle\psi_{111113}\rangle=\langle\psi_{111123}\rangle=\langle\psi_{111233}\rangle=0\,,\\
&\langle\psi_{111333}\rangle=\langle\psi_{112333}\rangle=\langle\psi_{133333}\rangle=\langle\psi_{233333}\rangle=0\,.
\end{aligned}\tag{2.27}$$

In order to break SO(3) to A_5, we introducing two driving fields $\psi^d_{\underline{1}}\sim\underline{1}$ and $\psi^d_{\underline{9}}\sim\underline{9}$, instead of $\underline{5}$. The flavon superpotential is constructed as

$$w_\psi=\psi^d_{\underline{1}}\left(\mu_\psi^2-c_{\rho1}(\psi\psi)_{\underline{1}}\right)+c_{\psi2}\psi^d_{\underline{9}}(\psi\psi)_{\underline{9}}\,.\tag{2.28}$$

Minimisation respect to the driving fields gives rise to

$$\frac{\partial w_\psi}{\partial\psi^d_{\underline{1}}}=\mu_\psi^2-c_{\psi1}(\psi\psi)_{\underline{1}}=0\,,\tag{2.29}$$

$$\frac{\partial w_\psi}{\partial\psi^d_{\underline{9}}}=c_{\psi2}(\psi\psi)_{\underline{9}}=0\,.\tag{2.30}$$

Taking eq. (2.27) to the above equations, we see that eq. (2.30) is automatically satisfied and eq. (2.29) leads to $\langle\psi_{111111}\rangle=\pm\mu_\psi/(4\sqrt{21c_{\psi1}})$.

The uniqueness of SO(3) $\to$ A_5. We vary ψ away from the A_5-invariant VEV, $\psi\to\langle\psi\rangle+\delta\psi$ and require that eqs. (2.29) and (2.30) are still satisfied. Then, we will get the constraints on $\delta\psi$,

$$\begin{aligned}
&\delta\psi_{111111}=\delta\psi_{111133}=\delta\psi_{113333}=\delta\psi_{333333}=0\,,\\
&\delta\psi_{111112}=\sqrt{5}b_2\delta\psi_{123333}\,, && \delta\psi_{111233}=b_1\psi_{123333}\,,\\
&\delta\psi_{111113}=-\frac{\sqrt{5}}{3}b_1\delta\psi_{111333}\,, && \delta\psi_{133333}=\frac{\sqrt{5}}{3}b_1\psi_{111333}\,,\\
&\delta\psi_{112333}=b_2\delta\psi_{111123}\,, && \delta\psi_{233333}=-\sqrt{5}b_1\psi_{111123}\,,
\end{aligned}\tag{2.31}$$

leaving also three unconstrained parameters $\delta\psi_{111123}$, $\delta\psi_{111333}$ and $\delta\psi_{123333}$. The unconstrained small parameters $\delta\psi$ can be rotated away if we consider a SO(3) basis transformation, $g_{\{\alpha^a\}}=\mathbb{1}_{3\times3}+\alpha^a\tau^a$ with $\alpha^1\to-4\sqrt{\frac{15}{7}}\delta\psi_{123333}/\mu_\psi$, $\alpha^2\to4\sqrt{\frac{5}{21}}\delta\psi_{111333}/\mu_\psi$, and $\alpha^3\to-4\sqrt{\frac{15}{7}}\delta\psi_{111123}/\mu_\psi$. Therefore, eqs. (2.29) and (2.30) uniquely break SO(3) to A_5.

2.3 Representation decomposition

After SO(3) is broken to a non-Abelian discrete group, it is necessary to decompose each irrep of SO(3) to a couple of irreps of the discrete one. This task is achieved by comparing reduction of Kronecker products of representations of SO(3) with those of the discrete one [51].

For irreps of SO(3) decomposed to irreps of A_4, we identify $\underline{1}$, $\underline{3}$ of SO(3) with **1**, **3** of A_4, respectively and compare the Kronecker products

$$\underline{3}\times\underline{3}=\underline{1}+\underline{3}+\underline{5}\tag{2.32}$$

in SO(3) with

$$\mathbf{3}\times\mathbf{3}=\mathbf{1}+\mathbf{1}'+\mathbf{1}''+\mathbf{3}+\mathbf{3} \tag{2.33}$$

in A_4. Since the right hand sides of both equations are identical, $\underline{5}$ of SO(3) is decomposed to $\mathbf{1}'+\mathbf{1}''+\mathbf{3}$ of A_4. One further compares right hand side of

$$\underline{3}\times\underline{5}=\underline{3}+\underline{5}+\underline{7} \tag{2.34}$$

with that of

$$\mathbf{3}\times(\mathbf{1}'+\mathbf{1}''+\mathbf{3})=\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'+\mathbf{1}''+\mathbf{3}+\mathbf{3} \tag{2.35}$$

and obtains $\underline{7}=\mathbf{1}+\mathbf{3}+\mathbf{3}$, where $\mathbf{1}'\times\mathbf{3}=\mathbf{3}$ and $\mathbf{1}''\times\mathbf{3}=\mathbf{3}$ are used. Continuing to play this game, we can get decomposition of as high irrep of SO(3) as we want into irreps of A_4.

This game is directly applied into irrep decomposition in S_4 and A_5. In S_4, there are five irreps: $\mathbf{1}$ (the trivial singlet), $\mathbf{1}'$ (different from $\mathbf{1}'$ of A_4), $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$. In A_5, there are five irreps: $\mathbf{1}$ (the trivial singlet), $\mathbf{3}$, $\mathbf{3}'$, $\mathbf{4}$ and $\mathbf{5}$. Keeping in mind the Kronecker products

$$\begin{aligned}&\mathbf{1}'\times\mathbf{1}'=\mathbf{1}\,, \qquad \mathbf{1}'\times\mathbf{2}=\mathbf{2}\,, \qquad \mathbf{2}\times\mathbf{2}=\mathbf{1}+\mathbf{1}'+\mathbf{2}\,,\\ &\mathbf{3}\times\mathbf{3}=\mathbf{3}'\times\mathbf{3}'=\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{3}'\,, \quad \mathbf{3}\times\mathbf{3}'=\mathbf{1}'+\mathbf{2}+\mathbf{3}+\mathbf{3}'\end{aligned} \tag{2.36}$$

in S_4, and

$$\begin{aligned}&\mathbf{3}\times\mathbf{3}=\mathbf{1}+\mathbf{3}+\mathbf{5}\,, \qquad \mathbf{3}'\times\mathbf{3}'=\mathbf{1}+\mathbf{3}'+\mathbf{5}\,, \qquad \mathbf{3}\times\mathbf{3}'=\mathbf{4}+\mathbf{5}\,,\\ &\mathbf{3}\times\mathbf{4}=\mathbf{3}'+\mathbf{4}+\mathbf{5}\,, \qquad \mathbf{3}'\times\mathbf{4}=\mathbf{3}+\mathbf{4}+\mathbf{5}\,, \qquad \mathbf{3}\times\mathbf{5}=\mathbf{3}'\times\mathbf{5}=\mathbf{3}+\mathbf{3}'+\mathbf{4}+\mathbf{5}\,,\\ &\mathbf{4}\times\mathbf{4}=\mathbf{1}+\mathbf{3}+\mathbf{3}'+\mathbf{4}+\mathbf{5}\,, \qquad \mathbf{4}\times\mathbf{5}=\mathbf{3}+\mathbf{3}'+\mathbf{3}'+\mathbf{4}+\mathbf{5}+\mathbf{5}\,,\\ &\mathbf{5}\times\mathbf{5}=\mathbf{1}+\mathbf{3}+\mathbf{3}'+\mathbf{4}+\mathbf{4}+\mathbf{5}+\mathbf{5}\end{aligned} \tag{2.37}$$

in A_5, and comparing them with Kronecker products in SO(3), we obtain irrep decompositions in S_4 and A_5, respectively.

We summarise decomposition of irreps of SO(3) (up to $\underline{13}$) to irreps of A_4, S_4 and A_5 in table 2.

Before ending this section, we show more details of how a irrep of SO(3) is decomposed into irreps of A_4 as follows, which will be useful for our discussion in the next two sections.

- For a triplet $\underline{3}$ of SO(3), $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T$, it is also a triplet $\mathbf{3}$ of A_4.
- A $\underline{5}$-plet of SO(3), χ, can be represented as a rank-2 tensor χ_{ij} in the 3d space. It is symmetric, $\chi_{ij}=\chi_{ji}$, and traceless, $\chi_{11}+\chi_{22}+\chi_{33}=0$. Independent components can be chosen as χ_{11}, χ_{12}, χ_{13}, χ_{23} and χ_{33}. The $\underline{5}$-plet is decomposed to two non-trivial singlets $\mathbf{1}'$ and $\mathbf{1}''$ and one triplet $\mathbf{3}$ of A_4. It is useful to re-parametrise χ in the form

$$\chi=\begin{pmatrix}\frac{1}{\sqrt{3}}(\chi'+\chi'') & \frac{1}{\sqrt{2}}\chi_3 & \frac{1}{\sqrt{2}}\chi_2\\ \frac{1}{\sqrt{2}}\chi_3 & \frac{1}{\sqrt{3}}(\omega\chi'+\omega^2\chi'') & \frac{1}{\sqrt{2}}\chi_1\\ \frac{1}{\sqrt{2}}\chi_2 & \frac{1}{\sqrt{2}}\chi_1 & \frac{1}{\sqrt{3}}(\omega^2\chi'+\omega\chi'')\end{pmatrix}, \tag{2.38}$$

SO(3)	A_4	S_4	A_5
$\underline{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\underline{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
$\underline{5}$	$\mathbf{1'}+\mathbf{1''}+\mathbf{3}$	$\mathbf{2}+\mathbf{3'}$	$\mathbf{5}$
$\underline{7}$	$\mathbf{1}+\mathbf{3}+\mathbf{3}$	$\mathbf{1'}+\mathbf{3}+\mathbf{3'}$	$\mathbf{3'}+\mathbf{4}$
$\underline{9}$	$\mathbf{1}+\mathbf{1'}+\mathbf{1''}+\mathbf{3}+\mathbf{3}$	$\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{3'}$	$\mathbf{4}+\mathbf{5}$
$\underline{11}$	$\mathbf{1'}+\mathbf{1''}+\mathbf{3}+\mathbf{3}+\mathbf{3}$	$\mathbf{2}+\mathbf{3}+\mathbf{3}+\mathbf{3'}$	$\mathbf{3}+\mathbf{3'}+\mathbf{5}$
$\underline{13}$	$\mathbf{1}+\mathbf{1}+\mathbf{1'}+\mathbf{1''}+\mathbf{3}+\mathbf{3}+\mathbf{3}$	$\mathbf{1}+\mathbf{1'}+\mathbf{2}+\mathbf{3}+\mathbf{3'}+\mathbf{3'}$	$\mathbf{1}+\mathbf{3}+\mathbf{4}+\mathbf{5}$

Table 2. Decomposition of some irreps of SO(3) into irreps of A_4, S_4 and A_5. Results of decomposition to irreps of A_4 have been given in [44].

where $\omega = e^{2i\pi/3}$. This parametrisation has two advantages. One is the simple transformation property in A_4,

$$\chi' \sim \mathbf{1}' , \quad \chi'' \sim \mathbf{1}'' , \quad \chi_{\mathbf{3}} \equiv (\chi_1, \chi_2, \chi_3) \sim \mathbf{3} \text{ of } A_4 . \tag{2.39}$$

The other is the normalised kinetic term,

$$\begin{aligned}(\partial_\mu \chi^* \partial^\mu \chi)_{\underline{1}} &= \partial_\mu \chi'^* \partial^\mu \chi' + \partial_\mu \chi''^* \partial^\mu \chi'' + \partial_\mu \chi_{\mathbf{3}}^\dagger \partial^\mu \chi_{\mathbf{3}} \\ &= \partial_\mu \chi'^* \partial^\mu \chi' + \partial_\mu \chi''^* \partial^\mu \chi'' + \partial_\mu \chi_1^* \partial^\mu \chi_1 + \partial_\mu \chi_2^* \partial^\mu \chi_2 + \partial_\mu \chi_3^* \partial^\mu \chi_3 .\end{aligned} \tag{2.40}$$

- The $\underline{7}$-plet of SO(3) is a symmetric and traceless rank-3 tensor in the 3d space. It is decomposed to one trivial singlet $\mathbf{1}$ and two triplets $\mathbf{3}$ of A_4. The former mentioned ξ can be re-labelled as

$$\begin{aligned}&\xi_{123} = \frac{1}{\sqrt{6}}\xi_0 , \\ &\xi_{111} = -\frac{2}{\sqrt{10}}\xi'_1 , \qquad \xi_{112} = \frac{1}{\sqrt{10}}\xi'_2 - \frac{1}{\sqrt{6}}\xi_2 , \qquad \xi_{113} = \frac{1}{\sqrt{10}}\xi'_3 + \frac{1}{\sqrt{6}}\xi_3 , \\ &\xi_{133} = \frac{1}{\sqrt{10}}\xi'_1 - \frac{1}{\sqrt{6}}\xi_1 , \qquad \xi_{233} = \frac{1}{\sqrt{10}}\xi'_2 + \frac{1}{\sqrt{6}}\xi_2 , \qquad \xi_{333} = -\frac{2}{\sqrt{10}}\xi'_3 .\end{aligned} \tag{2.41}$$

Here,

$$\xi_0 \sim \mathbf{1} , \quad \xi_{\mathbf{3}} \equiv (\xi_1, \xi_2, \xi_3) \sim \mathbf{3} , \quad \xi'_{\mathbf{3}} \equiv (\xi'_1, \xi'_2, \xi'_3) \sim \mathbf{3} \text{ of } A_4 . \tag{2.42}$$

And the kinetic term is also normalised,[2]

$$\begin{aligned}(\partial_\mu \xi^* \partial^\mu \xi)_{\underline{1}} &= \partial_\mu \xi_0^* \partial^\mu \xi_0 + \partial_\mu \xi_{\mathbf{3}}^\dagger \partial^\mu \xi_{\mathbf{3}} + \partial_\mu \xi'^\dagger_{\mathbf{3}} \partial^\mu \xi'_{\mathbf{3}} \\ &= \partial_\mu \xi_0^* \partial^\mu \xi_0 + \partial_\mu \xi_1^* \partial^\mu \xi_1 + \partial_\mu \xi_2^* \partial^\mu \xi_2 + \partial_\mu \xi_3^* \partial^\mu \xi_3 + \partial_\mu \xi'^*_1 \partial^\mu \xi'_1 \\ &\quad + \partial_\mu \xi'^*_2 \partial^\mu \xi'_2 + \partial_\mu \xi'^*_3 \partial^\mu \xi'_3 .\end{aligned} \tag{2.43}$$

Since ξ_0 is a trivial singlet of A_4, once ξ_0 gets a non-zero VEV, SO(3) will be broken but A_4 is still preserved. This is consistent with the discussion in the former subsection.

[2]Here we ignore the gauge interactions. Consequence of the gauge interactions will be given later in section 4.6.

3 The further breaking of A_4 to residual Z_3 and Z_2

In A_4 lepton flavour models, A_4 has to be broken to generate flavour mixing. In most of these models, residual symmetries Z_3 and Z_2 are preserved respectively in the charged lepton sector and neutrino sector after A_4 breaking. These residual symmetries are not precise but good approximate symmetries. The misalignment between Z_3 and Z_2 leading to a mixing with tri-bimaximal mixing pattern at leading order.

Embedding A_4 to the continuous SO(3) symmetry forces strong constraints on couplings, and the breaking of A_4 to Z_3 and Z_2 becomes very non-trivial. In this section, we will show, for definiteness, how to realise $A_4 \to Z_3$ and Z_2 in the framework of supersymmetric SO(3)-invariant theory.

3.1 $A_4 \to Z_3$

The breaking of A_4 to Z_3 can be simply realised by using a triplet $\underline{3}$ of SO(3). We denote such a flavon as φ. In order to obtain the Z_3-invariant VEV, we introduce an $\underline{1}$-plet driving field $\varphi^d_{\underline{1}}$ and a $\underline{5}$-plet driving field $\varphi^d_{\underline{5}}$ and consider the following SO(3)-invariant superpotential

$$w_\varphi = \varphi^d_{\underline{1}} \left(f_1(\varphi\varphi)_{\underline{1}} - \mu^2_\varphi\right) + \frac{f_2}{\Lambda}\left(\varphi^d_{\underline{5}}\left(\xi(\varphi\varphi)_{\underline{5}}\right)_{\underline{5}}\right)_{\underline{1}} . \tag{3.1}$$

Here as appearing in the non-renormalisable term, the scale Λ is assumed to be higher than the scale of SO(3) breaking to A_4.

Minimisation of the superpotential gives rise to

$$\begin{aligned}
\frac{\partial w_\varphi}{\partial \varphi^d_{\underline{1}}} &= f_1(\varphi\varphi)_{\underline{1}} - \mu^2_\varphi = 0\,,\\
\frac{\partial w_\varphi}{\partial \varphi^d_{\underline{5}}} &= \frac{f_2}{\Lambda}\left(\xi(\varphi\varphi)_{\underline{5}}\right)_{\underline{5}} = 0\,,
\end{aligned} \tag{3.2}$$

whose detailed formula is listed in appendix B. Starting from the A_4-invariant VEV $\langle\xi\rangle$ in eq. (2.9), we use $\left(\xi(\varphi\varphi)_{\underline{5}}\right)_{\underline{5}} = 0$ to derive $\varphi_1^2 = \varphi_2^2 = \varphi_3^2$, and $f_1(\varphi\varphi)_{\underline{1}} - \mu^2_\varphi = 0$ to determine the value of φ_1^2. Here, we directly write out the following complete list of solutions

$$\begin{pmatrix}\langle\varphi_1\rangle\\ \langle\varphi_2\rangle\\ \langle\varphi_3\rangle\end{pmatrix} = \pm v_\varphi \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix},\ \begin{pmatrix}1\\-1\\-1\end{pmatrix},\ \begin{pmatrix}-1\\1\\-1\end{pmatrix},\ \begin{pmatrix}-1\\-1\\1\end{pmatrix} \right\}, \tag{3.3}$$

where $v_\varphi = \mu_\varphi/\sqrt{3f_1}$. For non-zero v_φ, all four VEVs break the A_4 symmetry. Each VEV preserves a different Z_3 group. In detail, $(1,1,1)^T$ preserves $Z_3^t = \{\mathbf{1}, t, t^2\}$, $(1,-1,-1)^T$ preserves $Z_3^{sts} = \{\mathbf{1}, sts, (sts)^2\}$, $(-1,1,-1)^T$ preserves $Z_3^{st} = \{\mathbf{1}, st, (st)^2\}$, and $(-1,-1,1)^T$ preserves $Z_3^{ts} = \{\mathbf{1}, ts, (ts)^2\}$. These Z_3 groups are conjugate to each and have no physical difference [57, 58].

Eq. (3.2) involves interactions between φ and ξ, specifically the non-renormalisable term which results in the breaking of A_4. These terms may influence the VEV of ξ and shift it away from the A_4-invariant one. In general, this shifting effect is small enough due to suppression of the higher dimensional operator. In section 4, we will construct a flavour model, and based on the model, we will discuss in detail the shift of the ξ VEV due to non-normalisable interactions with the other flavons in section 4.4. As we will prove therein, the shift effect is suppressed by the scale Λ and in general very small.

3.2 $A_4 \to Z_2$

We use the $\underline{5}$-plet χ to achieve the $A_4 \to Z_2$ breaking. The relevant superpotential terms could be considered as follows

$$w_\chi = \chi^d_{\underline{1}}\left(g_1(\chi\chi)_{\underline{1}} - \mu^2_\chi\right) + \frac{g_2}{\Lambda}\left(\chi^d_{\underline{3}}\left(\xi(\chi\chi)_{\underline{5}}\right)_{\underline{3}}\right)_{\underline{1}} + \frac{g_3}{\Lambda}\left(\chi^d_{\underline{3}}\left(\xi(\chi\chi)_{\underline{9}}\right)_{\underline{3}}\right)_{\underline{1}} + g_4\left(\chi^d_{\underline{5}}(\chi\xi)_{\underline{5}}\right)_{\underline{1}}, \quad (3.4)$$

where the driving fields $\chi^d_{\underline{1}}$, $\chi^d_{\underline{3}}$ and $\chi^d_{\underline{5}}$ are $\underline{1}$-, $\underline{3}$- and $\underline{5}$-plets of SO(3). Minimisation of the superpotential results in equations

$$\begin{aligned}
\frac{\partial w_\chi}{\partial \chi^d_{\underline{1}}} &= g_1(\chi\chi)_{\underline{1}} - \mu^2_\chi = 0\,, \\
\frac{\partial w_\chi}{\partial \chi^d_{\underline{3}}} &= \frac{g_2}{\Lambda}\left(\xi(\chi\chi)_{\underline{5}}\right)_{\underline{3}} - \frac{g_3}{\Lambda}\left(\xi(\chi\chi)_{\underline{9}}\right)_{\underline{3}} = 0\,, \\
\frac{\partial w_\chi}{\partial \chi^d_{\underline{5}}} &= g_4(\chi\xi)_{\underline{5}} = 0\,.
\end{aligned} \quad (3.5)$$

Given the A_4-invariant VEV $\langle\xi\rangle$ in eq. (2.9) as input, $(\chi\xi)_{\underline{5}} = 0$ leads to $\chi' = \chi'' = 0$. Then, $\left(\xi(\chi\chi)_{\underline{5}}\right)_{\underline{3}}$ takes the same form as $\left(\xi(\chi\chi)_{\underline{9}}\right)_{\underline{3}}$ and the requirement $\left(\xi(\chi\chi)_{\underline{5}}\right)_{\underline{3}} = 0$ or $\left(\xi(\chi\chi)_{\underline{9}}\right)_{\underline{3}} = 0$ results in $\chi_1\chi_2 = \chi_2\chi_3 = \chi_3\chi_1 = 0$. Therefore, two of χ_1, χ_2 and χ_3 have to be zero. And the rest non-vanishing one is determined by $g_1(\chi\chi)_{\underline{1}} - \mu^2_\chi = 0$. We obtain the following complete list of solutions,

$$\begin{pmatrix} \langle\chi'\rangle \\ \langle\chi''\rangle \\ \begin{pmatrix} \langle\chi_1\rangle \\ \langle\chi_2\rangle \\ \langle\chi_3\rangle \end{pmatrix} \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \begin{pmatrix} \pm v_\chi \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \begin{pmatrix} 0 \\ \pm v_\chi \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \begin{pmatrix} 0 \\ 0 \\ \pm v_\chi \end{pmatrix} \end{pmatrix} \right\}, \quad (3.6)$$

where $v_\chi = \mu_\chi/\sqrt{g_1}$. These VEVs satisfy Z_2 symmetries. In details, the first, second, and third pairs preserve $Z_2^s = \{\mathbf{1}, s\}$, $Z_2^{tst^2} = \{\mathbf{1}, tst^2\}$, $Z_2^{t^2st} = \{\mathbf{1}, t^2st\}$, respectively. All these VEVs are conjugate with each other and have no physical differences [57, 58]. There is a new scale μ_χ introduced in the superpotential.

3.3 Spontaneously splitting $\mathbf{1}'$ with $\mathbf{1}''$ of A_4

In A_4 models, the three singlet irreps $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$ are usually assigned to e^c, μ^c and τ^c (or their permutation), respectively. These irreps are independent with each other in A_4 and the generated e, μ and τ masses are independent with each other.

In the framework of SO(3), the non-trivial singlet irreps $\mathbf{1}'$ and $\mathbf{1}''$ are obtained from the decomposition of $\underline{5}$ of SO(3) (or higher irreps, e.g., $\underline{9}$ etc), as shown in table 2. These singlets are always correlated with each other. As a consequence, if we directly arrange two of the charged leptons (e.g., μ^c and τ^c) to the same $\underline{5}$ of SO(3), we have to face a fine tuning of masses of these two charged leptons. In this subsection, we are going to consider how to avoid this problem from the spontaneous symmetry breaking of A_4.

We introduce another $\underline{5}$-plet flavon ζ,

$$\zeta = \begin{pmatrix} \frac{1}{\sqrt{3}}(\zeta' + \zeta'') & \frac{1}{\sqrt{2}}\zeta_3 & \frac{1}{\sqrt{2}}\zeta_2 \\ \frac{1}{\sqrt{2}}\zeta_3 & \frac{1}{\sqrt{3}}(\omega\zeta' + \omega^2\zeta'') & \frac{1}{\sqrt{2}}\zeta_1 \\ \frac{1}{\sqrt{2}}\zeta_2 & \frac{1}{\sqrt{2}}\zeta_1 & \frac{1}{\sqrt{3}}(\omega^2\zeta' + \omega\zeta'') \end{pmatrix}, \tag{3.7}$$

and three driving fields $\zeta^d_{\underline{1}}$, $\zeta^d_{\underline{3}}$ and $\tilde{\zeta}^d_{\underline{1}}$ with the following superpotential

$$w_\zeta = \zeta^d_{\underline{1}} \left(\frac{h_1}{\Lambda} \left(\zeta(\zeta\zeta)_{\underline{5}}\right)_{\underline{1}} - \mu^2_\zeta \right) + h_2 \left(\zeta^d_{\underline{3}}(\zeta\xi)_{\underline{3}}\right)_{\underline{1}} + h_3 \tilde{\zeta}^d_{\underline{1}}(\zeta\zeta)_{\underline{1}} \,. \tag{3.8}$$

Minimisaiton of the superpotential gives to

$$\begin{aligned} \frac{\partial w_\zeta}{\partial \zeta^d_{\underline{1}}} &= \frac{h_1}{\Lambda} \left(\zeta(\zeta\zeta)_{\underline{5}}\right)_{\underline{1}} - \mu^2_\zeta = 0 \\ \frac{\partial w_\zeta}{\partial \zeta^d_{\underline{3}}} &= h_2(\zeta\xi)_{\underline{3}} = 0 \\ \frac{\partial w_\zeta}{\partial \tilde{\zeta}^d_{\underline{1}}} &= h_3(\zeta\zeta)_{\underline{1}} = 0 \,. \end{aligned} \tag{3.9}$$

The second row directly determines $\zeta_1 = \zeta_2 = \zeta_3 = 0$. It leaves the third row simplified to $(\zeta\zeta)_{\underline{1}} = 2\zeta'\zeta'' = 0$, resulting in $\zeta'' = 0$ (or $\zeta' = 0$). The rest one, ζ' (or ζ''), is determined by the first row, which is simplified to $\frac{h_1}{\Lambda}(\zeta')^3 - \mu^2_\zeta = 0$, (or $\frac{h_1}{\Lambda}(\zeta'')^3 - \mu^2_\zeta = 0$). These results are summarised as

$$\begin{pmatrix} \langle\zeta'\rangle \\ \langle\zeta''\rangle \\ \begin{pmatrix} \langle\zeta_1\rangle \\ \langle\zeta_2\rangle \\ \langle\zeta_3\rangle \end{pmatrix} \end{pmatrix} = \left\{ \begin{pmatrix} v_\zeta\omega^i \\ 0 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} 0 \\ v_\zeta\omega^i \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \right\}, \tag{3.10}$$

with $v_\zeta = \sqrt[3]{\sqrt{3}\mu^2_\zeta\Lambda/(2h_1)}$ and $i = 0, 1, 2$. We will see how this VEV can separate μ and τ masses in the next section.

To summarise, we realise the breaking of A_4 to Z_3 and Z_2 and achieve to split $\mathbf{1}'$ with $\mathbf{1}''$ of A_4 based on SO(3)-invariant superpotential. The scales representing the breaking of A_4, v_φ, v_χ and v_ζ, should be much lower than the scale of SO(3) breaking v_ξ. This can be satisfied by treating μ^2_φ, μ^2_χ and μ^2_ζ as effective descriptions from higher dimensional operators. One may notice that there may exist some unnecessary interactions which are not written out but cannot be forbidden based on current field arrangements. A detailed

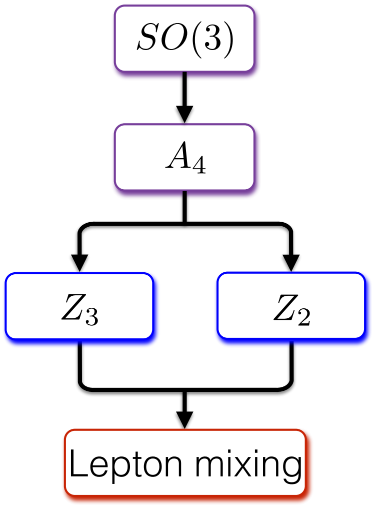

Figure 2. A sketch of the symmetry breaking in the model and how flavour mixing is generated. The flavour symmetry at high energy is assumed to be supersymmetric SO(3). It is broken first to A_4, which then breaks, at a lower scale, to the residual symmetry Z_3 in the charged lepton sector and Z_2 in the neutrino sector, with supersymmetry preserved throughout. The misalignment of the residual symmetries gives rise to flavour mixing.

discussion on how to forbid the unnecessary coupling will be given in the next section on the model building. Besides, the ways to realise $A_4 \to Z_3$, $A_4 \to Z_2$ and split $\mathbf{1}'$ from $\mathbf{1}''$ showing above are not the unique ways. One can introduce different irreps, combined with different driving fields to achieve them. This difference further leads to the difference of model building, which will not be discussed in this paper.

4 A supersymmetric A_4 model from SO(3) $\times$ U(1)

4.1 The model

In this section we will construct a supersymmetric A_4 model, based on SO(3) $\times$ U(1), with the breaking SO(3) $\to A_4$ and subsequently (at a lower scale) $A_4 \to Z_3, Z_2$, using the vacuum alignments discussed previously, where the misalignment of Z_3 in the charged lepton sector and Z_2 in the neutrino sector gives rise to lepton mixing. The model building strategy is shown in figure 2. The U(1) symmetry is used to forbid couplings which are unnecessary to generate the required flavon VEVs and flavour mixing. Note that no *ad hoc* discrete symmetries are introduced in this model.

In A_4 models, the right-handed charged leptons e^c, μ^c and τ^c are arranged as $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$ (or their permutation), respectively. In SO(3), the minimal irrep containing $\mathbf{1}'$ and $\mathbf{1}''$ is $\underline{5}$. In order to match with A_4 models, we embed $\mathbf{1}'$ and $\mathbf{1}''$ of A_4 to two different $\underline{5}$-plets of SO(3). In our model, we embed μ^c and τ^c to two different $\underline{5}$-plets R_μ and R_τ.[3] Four extra right-handed leptons are introduced for R_μ and R_τ, respectively. These particles should decouple at low energy theory to avoid unnecessary experimental constraints. We achieve

[3]Imbedding μ^c and τ^c into the same $\underline{5}$-plet leads to fine tuning between μ and τ masses.

JHEP11(2018)173

Fields	ℓ	N	e^c	R_μ	R_τ	$L_{\mu 0}$	$L_{\tau 0}$	L_μ	L_τ		
SO(3)	$\underline{3}$	$\underline{3}$	$\underline{5}$	$\underline{5}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{3}$	$\underline{3}$		
U(1)	$-\frac{2}{3}$	$+\frac{2}{3}$	$-\frac{7}{3}$	-1	$-\frac{1}{3}$	$+\frac{5}{6}$	0	$+\frac{2}{3}$	0		
Fields	η	$\bar{\eta}$	ξ	φ	χ	ζ	$H_{u,d}$				
SO(3)	$\underline{1}$	$\underline{1}$	$\underline{7}$	$\underline{3}$	$\underline{5}$	$\underline{5}$	$\underline{1}$				
U(1)	$+\frac{2}{3}$	$-\frac{2}{3}$	$+\frac{1}{3}$	$+1$	$-\frac{4}{3}$	$+\frac{1}{6}$	0				
Fields	$\eta^d_{\underline{1}}$	$\xi^d_{\underline{1}}$	$\xi^d_{\underline{5}}$	$\varphi^d_{\underline{1}}$	$\varphi^d_{\underline{5}}$	$\chi^d_{\underline{1}}$	$\chi^d_{\underline{3}}$	$\chi^d_{\underline{5}}$	$\zeta^d_{\underline{1}}$	$\zeta^d_{\underline{3}}$	$\tilde{\zeta}^d_{\underline{1}}$
SO(3)	$\underline{1}$	$\underline{1}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{1}$	$\underline{3}$	$\underline{5}$	$\underline{1}$	$\underline{3}$	$\underline{1}$
U(1)	0	$-\frac{2}{3}$	$-\frac{2}{3}$	-2	$-\frac{7}{3}$	$+2$	$+\frac{7}{3}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$

Table 3. Field arrangements in SO(3) $\times$ U(1) and decompositions of these fields in A_4 after SO(3) $\times$ U(1) is broken to A_4.

this goal by introducing two left-handed $\underline{3}$-plets L_μ, L_τ and two singlets $L_{\mu 0}$, $L_{\tau 0}$. We write out explicitly each components of the fermion multiplets in the 3d space as follows,

$$\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}, \quad N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}, \quad L_\mu = \begin{pmatrix} L_{\mu 1} \\ L_{\mu 2} \\ L_{\mu 3} \end{pmatrix}, \quad L_\tau = \begin{pmatrix} L_{\tau 1} \\ L_{\tau 2} \\ L_{\tau 3} \end{pmatrix},$$
$$R_\mu = \begin{pmatrix} \frac{1}{\sqrt{3}}(\mu^c + R''_\mu) & \frac{1}{\sqrt{2}} R_{\mu 3} & \frac{1}{\sqrt{2}} R_{\mu 2} \\ \frac{1}{\sqrt{2}} R_{\mu 3} & \frac{1}{\sqrt{3}}(\omega \mu^c + \omega^2 R''_\mu) & \frac{1}{\sqrt{2}} R_{\mu 1} \\ \frac{1}{\sqrt{2}} R_{\mu 2} & \frac{1}{\sqrt{2}} R_{\mu 1} & \frac{1}{\sqrt{3}}(\omega^2 \mu^c + \omega R''_\mu) \end{pmatrix},$$
$$R_\tau = \begin{pmatrix} \frac{1}{\sqrt{3}}(R'_\tau + \tau^c) & \frac{1}{\sqrt{2}} R_{\tau 3} & \frac{1}{\sqrt{2}} R_{\tau 2} \\ \frac{1}{\sqrt{2}} R_{\tau 3} & \frac{1}{\sqrt{3}}(\omega R'_\tau + \omega^2 \tau^c) & \frac{1}{\sqrt{2}} R_{\tau 1} \\ \frac{1}{\sqrt{2}} R_{\tau 2} & \frac{1}{\sqrt{2}} R_{\tau 1} & \frac{1}{\sqrt{3}}(\omega^2 R'_\tau + \omega \tau^c) \end{pmatrix}. \tag{4.1}$$

Here, $\ell_1 = (\nu_1, l_1)$, $\ell_2 = (\nu_2, l_2)$ and $\ell_3 = (\nu_3, l_3)$ are the three SM lepton doublets. $R_{\mu\mathbf{3}} \equiv (R_{\mu 1}, R_{\mu 2}, R_{\mu 3})^T$ and $R_{\tau\mathbf{3}} \equiv (R_{\tau 1}, R_{\tau 2}, R_{\tau 3})^T$ transform as $\mathbf{3}$ of A_4.

Charges for all relevant fields in SO(3) $\times$ U(1) are listed in table 3. Besides SO(3), we introduce additional U(1) symmetry to forbid unnecessary couplings.

4.2 Vacuum alignments

Terms leading to SO(3) breaking and A_4 breaking in the superpotential involving flavons and driving fields are given by

$$\begin{aligned} w_f \supset\ & \eta^d_{\underline{1}}\left(d_1 \eta\bar{\eta} - \mu_\eta^2\right) + \xi^d_{\underline{1}}\left(c_1 (\xi\xi)_{\underline{1}} - A_\xi \eta\right) + c_2 \left(\xi^d_{\underline{5}} (\xi\xi)_{\underline{5}}\right)_{\underline{1}} \\ & + \varphi^d_{\underline{1}} \left(f_1 (\varphi\varphi)_{\underline{1}} - \frac{f_\varphi}{\Lambda}\eta^3 \right) + \frac{f_2}{\Lambda}\left(\varphi^d_{\underline{5}} \left(\xi (\varphi\varphi)_{\underline{5}}\right)_{\underline{5}} \right)_{\underline{1}} \\ & + \chi^d_{\underline{1}} \left(\frac{g'_1}{\Lambda} (\chi\chi)_{\underline{1}} \eta - \frac{g_\chi}{\Lambda} \bar{\eta}^3 \right) + \frac{g_2}{\Lambda} \left(\chi^d_{\underline{3}} \left(\xi (\chi\chi)_{\underline{5}}\right)_{\underline{3}} \right)_{\underline{1}} + \frac{g_3}{\Lambda} \left(\chi^d_{\underline{3}} \left(\xi (\chi\chi)_{\underline{9}}\right)_{\underline{3}} \right)_{\underline{1}} + g_4 \left(\chi^d_{\underline{5}} (\chi\xi)_{\underline{5}} \right)_{\underline{1}} \\ & + \tilde{\zeta}^d_{\underline{1}} \left(\frac{h_1}{\Lambda} \left(\zeta (\zeta\zeta)_{\underline{5}}\right)_{\underline{1}} - \frac{h_\zeta}{\Lambda^2} \left(\zeta (\varphi\chi)_{\underline{5}}\right)_{\underline{1}} \eta \right) + h_2 \left(\zeta^d_{\underline{3}} (\zeta\xi)_{\underline{3}} \right)_{\underline{1}} + h_3 \tilde{\zeta}^d_{\underline{1}} (\zeta\zeta)_{\underline{1}} + \cdots . \end{aligned} \tag{4.2}$$

Here, the dots represent subleading corrections, which will be discussed in section 4.4. Compared with the superpotential terms in sections 2 and 3, eq. (4.2) takes a very similar form except the following differences:

- The constant μ_ξ^2 is not explicitly written out, but replaced by $A_\xi \eta$. Here η and $\bar{\eta}$ are SO(3) singlets. From the minimisation $\partial w_f / \partial \eta^d_{\underline{1}} = 0$, we know that both $\langle \eta \rangle$ and $\langle \bar{\eta} \rangle$ cannot be zero, and thus, we denote them as v_η and $v_{\bar{\eta}}$, respectively. Once η gets this VEV, $\mu_\xi^2 = A_\xi v_\eta$ is effectively obtained. This treatment is helpful for us to arrange charges for ξ. Otherwise only a Z_2 charge can be arranged for ξ.

- The constants μ_φ^2, μ_χ^2 and μ_ζ^2 are replaced by $f_\varphi \eta^3/\Lambda$, $g_\chi \bar{\eta}^3/\Lambda$ and $h_\zeta \left(\zeta(\varphi\chi)_{\underline{5}}\right)_{\underline{1}} \eta/\Lambda^2$, respectively. These constants are just effective description of the higher dimensional operators after the relevant flavons get VEVs,
$$\mu_\varphi^2 = \frac{f_\varphi}{\Lambda} v_\eta^3 \,, \quad \mu_\chi^2 = \frac{g_\chi}{\Lambda} v_{\bar{\eta}}^3 \,, \quad \mu_\zeta^2 = -i\sqrt{2} \frac{h_\zeta}{\Lambda^2} v_\zeta v_\varphi v_\chi v_\eta \,. \tag{4.3}$$

- The term $g_1 \chi^d_{\underline{1}} (\chi\chi)_{\underline{1}}$ is not explicitly written out, but effectively obtained from the operator $\frac{g_1'}{\Lambda} \chi^d_{\underline{1}} (\chi\chi)_{\underline{1}} \eta$ after η gains the VEV. In this case, g_1 is effectively expressed as $g_1 = g_1' v_\eta / \Lambda$. The term $\left(\varphi^d_{\underline{5}} \left(\xi(\varphi\varphi)_{\underline{5}}\right)_{\underline{5}}\right)_{\underline{1}}$ does not contribute since $(\varphi\varphi)_{\underline{5}}$ vanishes at the A_4-invariant VEV.

The approach for how the flavons obtained the required VEVs have been discussed in the former section. We do not repeat the relevant discussion here but just list the achieved VEVs of flavons,

$$\begin{aligned}
\xi^{A_4} &: \quad \langle \xi_{123} \rangle \equiv \frac{v_\xi}{\sqrt{6}} \,, \quad \langle \xi_{111} \rangle = \langle \xi_{112} \rangle = \langle \xi_{113} \rangle = \langle \xi_{133} \rangle = \langle \xi_{233} \rangle = \langle \xi_{333} \rangle = 0 \,; \\
\varphi^{Z_3} &: \quad \begin{pmatrix} \langle \varphi_1 \rangle \\ \langle \varphi_2 \rangle \\ \langle \varphi_3 \rangle \end{pmatrix} = v_\varphi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,; \\
\chi^{Z_2} &: \quad \begin{pmatrix} \langle \chi' \rangle \\ \langle \chi'' \rangle \\ \begin{pmatrix} \langle \chi_1 \rangle \\ \langle \chi_2 \rangle \\ \langle \chi_3 \rangle \end{pmatrix} \end{pmatrix} = v_\chi \begin{pmatrix} 0 \\ 0 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \,; \\
\zeta^{\mathbf{1}'} &: \quad \begin{pmatrix} \langle \zeta' \rangle \\ \langle \zeta'' \rangle \\ \begin{pmatrix} \langle \zeta_1 \rangle \\ \langle \zeta_2 \rangle \\ \langle \zeta_3 \rangle \end{pmatrix} \end{pmatrix} = v_\zeta \begin{pmatrix} 1 \\ 0 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \,,
\end{aligned} \tag{4.4}$$

where v_ξ, v_φ, v_χ and v_ζ are respectively given by

$$v_\xi = \sqrt{\frac{A_\xi v_\eta}{c_1}} \,, \quad v_\varphi = v_\eta \sqrt{\frac{f_\varphi v_\eta}{3 f_1 \Lambda}} \,, \quad v_\chi = v_{\bar{\eta}} \sqrt{\frac{g_\chi v_{\bar{\eta}}}{g_1' v_\eta}} \,, \quad v_\zeta = v_\eta \left(\frac{-f_\varphi g_\chi h_\zeta^2 v_{\bar{\eta}}^3}{2 f_1 g_1' h_1^2 \Lambda^3} \right)^{\frac{1}{4}} \,. \tag{4.5}$$

We briefly discuss the scales involved in the model. The VEV v_ξ represents the scale of SO(3) $\to A_4$ and v_φ and v_χ represent the scales of $A_4 \to Z_3$ and Z_2, respectively. VEVs of η and $\bar{\eta}$ do not break any non-Abelian symmetries but U(1), their role is to connect the scales of SO(3) breaking and A_4 breakings. For the scale of $A_4 \to Z_3$, $v_\varphi \ll v_\eta$ is naturally achieved due to the suppression of Λ in the dominator in eq. (4.5). For the scale of $A_4 \to Z_2$, $v_\chi \ll v_\eta$ can be achieved by either assuming a hierarchy $v_\eta \ll v_{\bar{\eta}}$ all assuming a small coefficient g_χ. The VEV v_ξ can be much larger than v_η and $v_{\bar{\eta}}$ if the dimension one parameter A_ξ is large enough. With the above treatment (but not the unique treatment), we can easily achieve a hierarchy of energy scales

$$\text{UV scale } (\Lambda) \gg \text{ scale of SO(3)} \to A_4\ (v_\xi) \ \gg \ \text{ scales of } A_4 \to Z_3, Z_2\ (v_\varphi, v_\chi)\,. \tag{4.6}$$

In the following, we simplify our discussion by assuming all dimensionless parameters in the flavon superpotential being of order one. In this case, orders of magnitude of v_ξ, v_φ, v_χ and v_ζ are determined by Λ, A_ξ, $v_{\bar{\eta}}$ and v_η as

$$v_\xi \sim v_\eta \sqrt{\frac{A_\xi}{v_\eta}}\,, \quad v_\varphi \sim v_\eta \sqrt{\frac{v_\eta}{\Lambda}}\,, \quad v_\chi \sim v_{\bar{\eta}} \sqrt{\frac{v_{\bar{\eta}}}{v_\eta}}\,, \quad v_\zeta \sim v_\eta \left(\frac{v_{\bar{\eta}}}{\Lambda}\right)^{\frac{3}{4}}\,. \tag{4.7}$$

The hierarchy in eq. (4.6) is obtained by requiring $\Lambda \gg A_\xi \gg v_{\bar{\eta}} \gg v_\eta$.

4.3 Lepton masses

Lagrangian terms for generating charged lepton masses are given by

$$w_\ell = w_{e^c} + w_{R_\mu} + w_{R_\tau} + w_N \tag{4.8}$$

with

$$\begin{aligned}
w_{e^c} &\supset \frac{y_{e1}}{\Lambda^3}(\varphi\varphi)_{\underline{1}}(\varphi\ell)_{\underline{1}} e^c H_d + \frac{y_{e2}}{\Lambda^3}\Big(((\varphi\varphi)_{\underline{5}}\varphi)_{\underline{3}}\ell\Big)_{\underline{1}} e^c H_d\,,\\
w_{R_\mu} &\supset \frac{y_{\mu1}}{\Lambda^2}\big(\varphi(\ell R_\mu)_{\underline{3}}\big)_{\underline{1}}\bar{\eta} H_d + \frac{y_{\mu2}}{\Lambda}\big(\varphi(L_\mu R_\mu)_{\underline{3}}\big)_{\underline{1}}\bar{\eta} + Y_{\mu1} L_{\mu0}\,(\zeta R_\mu)_{\underline{1}}\\
&\quad + \frac{Y_{\mu3}}{\Lambda}\big(\xi(\ell R_\mu)_{\underline{7}}\big)_{\underline{1}} H_d + Y_{\mu2}\big(\xi(L_\mu R_\mu)_{\underline{5}}\big)_{\underline{1}}\,,\\
w_{R_\tau} &\supset \frac{y_\tau}{\Lambda}\big(\varphi(\ell R_\tau)_{\underline{3}}\big)_{\underline{1}} H_d + \frac{Y_{\tau1}}{\Lambda} L_{\tau0}\big((\zeta\zeta)_{\underline{5}} R_\tau\big)_{\underline{1}} + Y_{\tau2}\big(\xi(L_\tau R_\tau)_{\underline{5}}\big)_{\underline{1}}\,,\\
w_N &\supset y_N(\ell N)_{\underline{1}} H_u + \frac{\lambda_\eta}{\Lambda}\bar{\eta}^2 (NN)_{\underline{1}} + \lambda_\chi\big(\chi(NN)_{\underline{5}}\big)_{\underline{1}}
\end{aligned} \tag{4.9}$$

at leading order. After the flavons get their VEVs, we arrive at the effective Yukawa couplings for leptons and Majorana masses for right-handed neutrinos.

The Yukawa coupling of e^c is given by

$$w_e^{\rm eff} = y_e \frac{v_\varphi^3}{\Lambda^3}\ \ell^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^c H_d \tag{4.10}$$

where $y_e = 3y_{e1} + 4y_{e2}$.

Couplings involving R_μ are given by

$$w_{R_\mu}^{\text{eff}} = (\ell^T, L_{\mu 0}, L_{\mu\mathbf{3}}^T) \begin{pmatrix} y_{\mu 1}\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda^2} V_\omega H_d & y_{\mu 1}\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda^2} V_\omega^* H_d & 2\sqrt{3} Y_{\mu 3}\frac{v_\xi}{\Lambda}\ \mathbb{1}_{3\times 3}\ H_d \\ 0 & Y_{\mu 1} v_\zeta & \mathbb{0}_{1\times 3} \\ y_{\mu 2}\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda} V_\omega & y_{\mu 2}\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda} V_\omega^* & 2\sqrt{3} Y_{\mu 2} v_\xi\ \mathbb{1}_{3\times 3} \end{pmatrix} \begin{pmatrix} \mu^c \\ R_\mu'' \\ R_{\mu\mathbf{3}} \end{pmatrix}, \tag{4.11}$$

where

$$V_\omega = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad V_\omega^* = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}. \tag{4.12}$$

$L_{\mu\mathbf{3}}$ and $R_{\mu\mathbf{3}}$ obtain three degenerate heavy masses $2\sqrt{3}Y_{\mu 2}v_\xi$. These mass are much heavier than the electroweak scale, and thus for the low energy theory, $L_{\mu\mathbf{3}}$ and $R_{\mu\mathbf{3}}$ decouple. $L_{\mu 0}$ and R''_μ obtain a mass $Y_{\mu 1}v_\zeta$. For v_ζ heavier than the electroweak scale, R''_μ decouples from the low energy theory. In this way, we successfully split R''_μ with μ^c. After the heavy leptons are integrated out, we are left with the following couplings at the low energy theory,

$$w_\mu^{\text{eff}} = y_\mu \frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda^2}\ \ell^T \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \mu^c H_d\,, \tag{4.13}$$

where $y_\mu = y_{\mu 1} - y_{\mu 2}Y_{\mu 3}/Y_{\mu 2}$ with the $y_{\mu 2}$ term obtained via a seesaw-like formula.

Those coupling to R_τ are given by

$$w_{R_\tau}^{\text{eff}} = (\ell^T, L_{\tau 0}, L_{\tau\mathbf{3}}^T) \begin{pmatrix} y_\tau\frac{v_\varphi}{\sqrt{3}\Lambda} V_\omega^* H_d & y_\tau\frac{v_\varphi}{\sqrt{3}\Lambda} V_\omega H_d & \mathcal{O}(y_\tau\frac{v_\varphi}{\sqrt{3}\Lambda}) H_d \\ 0 & Y_{\tau 1}\frac{2v_\zeta^2}{\sqrt{3}\Lambda} & \mathbb{0}_{1\times 3} \\ \mathbb{0}_{3\times 1} & \mathbb{0}_{3\times 1} & 2\sqrt{3}Y_{\tau 2}v_\xi\ \mathbb{1}_{3\times 3} \end{pmatrix} \begin{pmatrix} \tau^c \\ R_\tau' \\ R_{\tau\mathbf{3}} \end{pmatrix}. \tag{4.14}$$

$L_{\tau\mathbf{3}}$ and $R_{\tau\mathbf{3}}$ obtain three degenerate heavy masses $2\sqrt{3}Y_{\tau 2}v_\xi$, which are much heavier than the electroweak scale. $L_{\tau 0}$ and R'_τ obtain a mass $2Y_{\tau 1}v_\zeta^2/(\sqrt{3}\Lambda)$. This mass term should also be heavier than the electroweak scale such that R'_τ can decouple from the low energy theory. This mass term aims to split R'_τ with τ^c and it provides a stronger constraint to the scale v_ζ than that splitting R''_μ with μ^c. After all these heavy particles decouple, we obtain Yukawa coupling for τ^c at low energy as

$$w_\tau^{\text{eff}} = y_\tau \frac{v_\varphi}{\sqrt{3}\Lambda}\ \ell^T \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \tau^c H_d\,. \tag{4.15}$$

After the Higgs H_d gets the VEV $\langle H_d \rangle = v_d/\sqrt{2}$, we arrive at the charged lepton mass matrix

$$M_l = \begin{pmatrix} y_e\frac{v_\varphi^3}{\Lambda^3} & y_\mu\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda^2} & y_\tau\frac{v_\varphi}{\sqrt{3}\Lambda} \\ y_e\frac{v_\varphi^3}{\Lambda^3} & \omega y_\mu\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda^2} & \omega^2 y_\tau\frac{v_\varphi}{\sqrt{3}\Lambda} \\ y_e\frac{v_\varphi^3}{\Lambda^3} & \omega^2 y_\mu\frac{v_\varphi v_{\bar{\eta}}}{\sqrt{3}\Lambda^2} & \omega y_\tau\frac{v_\varphi}{\sqrt{3}\Lambda} \end{pmatrix} \frac{v_d}{\sqrt{2}} \tag{4.16}$$

in the basis ℓ and $(e^c, \mu^c, \tau^c)^T$. This matrix is diagonal by a unitary matrix U_l via $U_l^T M_l = \text{diag}\{m_e, m_\mu, m_\tau\}$ with

$$U_l = U_\omega \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \tag{4.17}$$

and

$$m_e = \left| \sqrt{3} y_e \frac{v_\varphi^3 v_d}{\sqrt{2}\Lambda^3} \right| \quad m_\mu = \left| y_\mu \frac{v_\varphi v_{\bar{\eta}} v_d}{\sqrt{2}\Lambda^2} \right| \quad m_\tau = \left| y_\tau \frac{v_\varphi v_d}{\sqrt{2}\Lambda} \right| . \tag{4.18}$$

In this model, since y_e, y_μ, and y_τ are totally independent free parameters, there is no need to introduce a fine tuning of them to fit the hierarchy of e, μ and τ masses.

If we naturally assume the dimensionless parameters y_e, y_μ and y_τ are all of order one, we then obtain $m_e : m_\mu : m_\tau \sim (v_\varphi/\Lambda)^2 : v_{\bar{\eta}}/\Lambda : 1$. The mass between $L_{\tau 0}$ and R'_τ is of order v_ζ^2/Λ. It should be much heavier than the electroweak scale to avoid the constraints from collider searches, i.e., $v_\zeta^2/v_\varphi \gg v_d$.

The realisation of neutrino masses is straightforward. The relevant superpotential terms at leading order are given by

$$w_N = y_N (\ell N)_{\underline{1}} H_u + \frac{\lambda_\eta}{\Lambda} \bar{\eta}^2 (NN)_{\underline{1}} + \lambda_\chi \big(\chi (NN)_{\underline{5}}\big)_{\underline{1}} . \tag{4.19}$$

The generated Dirac mass matrix between ν and N and Majorana mass matrix for N, in the bases $(\nu_1, \nu_2, \nu_3)^T$ and $(N_1, N_2, N_3)^T$, are respectively given by

$$\begin{aligned} M_D &= \frac{y_D v_u}{\sqrt{2}} \mathbb{1}_{3\times 3} , \\ M_M &= \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} , \end{aligned} \tag{4.20}$$

where $a = 2\lambda_{\bar{\eta}} v_{\bar{\eta}}^2/\Lambda$ and $b = 2\sqrt{2}\lambda_\chi v_\chi$. It is straightforward to diagonalise M_M via $U_\nu^\dagger M_M U_\nu^* = \text{diag}\{M_1, M_2, M_3\}$ with

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \end{pmatrix} P_\nu , \tag{4.21}$$

where $P_\nu = \text{diag}\{ e^{i\frac{\beta_1}{2}}, e^{i\frac{\beta_2}{2}}, e^{i\frac{\beta_3}{2}} \}$ and

$$M_1 = |a+b| , \qquad M_2 = |a| , \qquad M_3 = |a-b| , \tag{4.22}$$

$$\beta_1 = \arg(b+a) , \quad \beta_2 = \arg(a) , \quad \beta_3 = \arg(b-a) . \tag{4.23}$$

Applying the seesaw mechanism, $M_\nu = -M_D M_M^{-1} M_D^T$, we obtain that M_ν is diagonalised as $U_\nu^T M_\nu U_\nu = \text{diag}\{ m_1, m_2, m_3 \}$. The three mass eigenvalues for light neutrinos are given by $m_1 = y_D^2 v_u^2/(2|b+a|)$, $m_2 = y_D^2 v_u^2/(2|a|)$ and $m_3 = y_D^2 v_u^2/(2|b-a|)$. The PMNS matrix is given by $U_{\rm PMNS} = U_l^\dagger U_\nu = U_{\rm TBM} P_\nu$. We are left with the tri-bimaximal mixing.

In the model discussed so far, the crucial point in deriving an A_4-invariant VEV is the requirement $(\xi\xi)_{\underline{5}} = 0$, while those to derive a Z_2- or Z_3-invariant vacuum is the requirement $\big(\xi(\varphi\varphi)_{\underline{5}}\big)_{\underline{5}} = 0$ or $(\chi\xi)_{\underline{5}} = \big(\xi(\chi\chi)_{\underline{5}}\big)_{\underline{3}} = 0$, respectively. These requirements are obtained via the minimisation of the superpotential. However, extra terms may be involved in the superpotential and lead to that the above requirements do not hold explicitly. As a consequence, the relevant vacuums do not preserve the symmetries explicitly. In the next subsection, we will prove that after including these terms, the flavon VEVs do deviate from the former symmetric ones, but the size of the deviations are safely very small. Then in the subsequent subsection we consider subleading effects to the flavour mixing and show that it gives important corrections.

4.4 Subleading corrections to the vacuum (are negligible)

We first list terms in the flavon superpotential which cannot be avoided by the flavour symmetry SO(3) × U(1). The full flavon superpotential should be given by

$$w_f = w_f^{d\leqslant 3} + w_f^{d=4} + w_f^{d=5} + \cdots\,. \tag{4.24}$$

$w_f^{d\leqslant 3}$ represents renormalisable terms in the superpotential, and $w_f^{d=4}$ and $w_f^{d=5}$ are non-renormalisable quartic and quintic couplings, respectively. Up to quintic couplings, all terms are listed in table 4, classified by the driving fields. As mentioned above, we follow the general arrangement in most supersymmetric models that driving fields always linearly couple to flavon fields. Compared with eq. (4.2), a lot of new terms appear here. We will discuss how they modify the VEVs of ξ, φ, χ and ζ in detail.

First for ξ, couplings involving the driving field $\xi^d_{\underline{5}}$ include not just the renormalisable term $\Big(\xi^d_{\underline{5}}(\xi\xi)_{\underline{5}}\Big)_{\underline{1}}$, but also the quartic term $\Big(\xi^d_{\underline{5}}(\xi\varphi)_{\underline{5}}\Big)_{\underline{1}}\bar\eta$ and the quintic term $\Big(\xi^d_{\underline{5}}\chi\Big)_{\underline{1}}\eta^3$, $\Big(\xi^d_{\underline{5}}(\xi\xi)_{\underline{5}}\Big)_{\underline{1}}\eta\bar\eta$, etc. The minimisation $\partial w_f/\partial\xi^d_{\underline{5}} = 0$ does not lead to $(\xi\xi)_{\underline{5}} = 0$, but

$$(\xi\xi)_{\underline{5}} = \frac{1}{\Lambda}(\xi\varphi)_{\underline{5}}\,\bar\eta + \frac{1}{\Lambda}\chi(\varphi\varphi)_{\underline{1}} + \frac{1}{\Lambda^2}\chi\eta^3 + \frac{1}{\Lambda^2}(\xi\xi)_{\underline{5}}\eta\bar\eta + \cdots\,, \tag{4.25}$$

where the dots represent contribution of all rest terms involving $\xi^d_{\underline{5}}$ in table 4. Dimensionless free parameters are omitted here and in the following. Couplings involving the driving field $\xi^d_{\underline{1}}$ is modified into

$$(\xi\xi)_{\underline{1}} - A_\xi\eta = \frac{1}{\Lambda}\eta^2\bar\eta + \frac{1}{\Lambda^2}(\xi\xi)_{\underline{1}}\eta\bar\eta + \cdots\,, \tag{4.26}$$

where the dots represent contributions of the rest terms in table 4. We denote the shifted VEV as

$$\xi^{A_4} + \delta_\xi\,, \tag{4.27}$$

	renormalisable terms $w_f^{d\leqslant 3}$	quartic terms $w_f^{d=4}\times\Lambda$	quintic terms $w_f^{d=5}\times\Lambda^2$
$\eta^d_{\underline{1}}$	$\mu_\eta^2\eta^d_{\underline{1}}$, $\eta^d_{\underline{1}}\eta\bar\eta$	$\eta^d_{\underline{1}}(\xi\xi)_{\underline{1}}\bar\eta$, $\eta^d_{\underline{1}}\big((\varphi\chi)_{\underline{7}}\xi\big)_{\underline{1}}$	$\eta^d_{\underline{1}}\eta^2\bar\eta^2$, $\eta^d_{\underline{1}}\big((\xi\xi)_{\underline{5}}\chi\big)_{\underline{1}}\eta$, $\eta^d_{\underline{1}}\big((\varphi\varphi)_{\underline{5}}\chi\big)_{\underline{1}}\bar\eta$, $\eta^d_{\underline{1}}\big((\zeta\zeta)_{\underline{5}}(\varphi\chi)_{\underline{5}}\big)_{\underline{1}}$
$\xi^d_{\underline{1}}$	$A_\xi\xi^d_{\underline{1}}\eta$, $\xi^d_{\underline{1}}(\xi\xi)_{\underline{1}}$	$\xi^d_{\underline{1}}\eta^2\bar\eta$, $\eta^d_{\underline{1}}\big((\varphi\varphi)_{\underline{5}}\chi\big)_{\underline{1}}$	$\xi^d_{\underline{1}}(\xi\xi)_{\underline{1}}\eta\bar\eta$, $\xi^d_{\underline{1}}\big((\varphi\chi)_{\underline{7}}\xi\big)_{\underline{1}}\eta$, $\xi^d_{\underline{1}}(\varphi\varphi)_{\underline{1}}\bar\eta^2$, $\xi^d_{\underline{1}}\big((\zeta\zeta)_{\underline{1}}\big)^2$, $\xi^d_{\underline{1}}\left((\zeta\zeta)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\right)_{\underline{1}}$, $\xi^d_{\underline{1}}\left((\zeta\zeta)_{\underline{9}}(\zeta\zeta)_{\underline{9}}\right)_{\underline{1}}$
$\xi^d_{\underline{5}}$	$\big(\xi^d_{\underline{5}}(\xi\xi)_{\underline{5}}\big)_{\underline{1}}$	$\Big(\xi^d_{\underline{5}}(\xi\varphi)_{\underline{5}}\Big)_{\underline{1}}\bar\eta$, $(\xi^d_{\underline{5}}\chi)_{\underline{1}}(\varphi\varphi)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big(\xi(\zeta\zeta)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big(\xi(\zeta\zeta)_{\underline{9}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big(\chi(\varphi\varphi)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$	$(\xi^d_{\underline{5}}\chi)_{\underline{1}}\eta^3$, $\big(\xi^d_{\underline{5}}(\xi\xi)_{\underline{5}}\big)_{\underline{1}}\eta\bar\eta$, $\Big(\xi^d_{\underline{5}}\big(\xi(\varphi\chi)_{\underline{3}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big(\xi(\varphi\chi)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big(\xi(\varphi\chi)_{\underline{7}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}(\varphi\varphi)_{\underline{5}}\Big)_{\underline{1}}\bar\eta^2$, $\Big(\xi^d_{\underline{5}}\big(\varphi(\zeta\zeta)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}\bar\eta$, $\big(\xi^d_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}}(\zeta\zeta)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big((\zeta\zeta)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big((\zeta\zeta)_{\underline{5}}(\zeta\zeta)_{\underline{9}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\xi^d_{\underline{5}}\big((\zeta\zeta)_{\underline{9}}(\zeta\zeta)_{\underline{9}}\big)_{\underline{5}}\Big)_{\underline{1}}$
$\varphi^d_{\underline{1}}$	$\varphi^d_{\underline{1}}(\varphi\varphi)_{\underline{1}}$	$\varphi^d_{\underline{1}}\eta^3$	$\varphi^d_{\underline{1}}(\xi\xi)_{\underline{1}}\eta^2$, $\varphi^d_{\underline{1}}(\varphi\varphi)_{\underline{1}}\eta\bar\eta$, $\varphi^d_{\underline{1}}\left((\xi\xi)_{\underline{5}}(\xi\varphi)_{\underline{5}}\right)_{\underline{1}}$, $\varphi^d_{\underline{1}}\left((\xi\xi)_{\underline{9}}(\xi\varphi)_{\underline{9}}\right)_{\underline{1}}$
$\varphi^d_{\underline{5}}$	0	$\Big(\varphi^d_{\underline{5}}\big(\xi(\varphi\varphi)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$	$\Big(\varphi^d_{\underline{5}}\big((\xi\xi)_{\underline{5}}\varphi\big)_{\underline{5}}\Big)_{\underline{1}}\eta$, $\big(\varphi^d_{\underline{5}}(\varphi\varphi)_{\underline{5}}\big)_{\underline{1}}(\zeta\zeta)_{\underline{1}}$, $\big(\varphi^d_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}}(\varphi\varphi)_{\underline{1}}$, $\Big(\varphi^d_{\underline{5}}\big((\varphi\varphi)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\varphi^d_{\underline{5}}\big((\varphi\varphi)_{\underline{5}}(\zeta\zeta)_{\underline{9}}\big)_{\underline{5}}\Big)_{\underline{1}}$,
$\chi^d_{\underline{1}}$	0	$\chi^d_{\underline{1}}(\chi\chi)_{\underline{1}}\eta$, $\chi^d_{\underline{1}}\bar\eta^3$,	$\chi^d_{\underline{1}}(\xi\xi)_{\underline{1}}(\chi\chi)_{\underline{1}}$, $\chi^d_{\underline{1}}\left((\xi\xi)_{\underline{5}}(\chi\chi)_{\underline{5}}\right)_{\underline{1}}$, $\chi^d_{\underline{1}}\left((\xi\xi)_{\underline{9}}(\chi\chi)_{\underline{9}}\right)_{\underline{1}}$
$\chi^d_{\underline{3}}$	0	$\Big(\chi^d_{\underline{3}}\big(\xi(\chi\chi)_{\underline{5}}\big)_{\underline{3}}\Big)_{\underline{1}}$, $\Big(\chi^d_{\underline{3}}\big(\xi(\chi\chi)_{\underline{9}}\big)_{\underline{3}}\Big)_{\underline{1}}$	$\big(\chi^d_{\underline{3}}(\xi\chi)_{\underline{3}}\big)_{\underline{1}}\bar\eta^2$, $(\chi^d_{\underline{3}}\varphi)_{\underline{1}}(\chi\chi)_{\underline{1}}\bar\eta$, $\Big(\chi^d_{\underline{3}}\big(\varphi(\chi\chi)_{\underline{5}}\big)_{\underline{3}}\Big)_{\underline{1}}\bar\eta$, $\Big(\chi^d_{\underline{3}}\big((\chi\chi)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{3}}\Big)_{\underline{1}}$,
$\chi^d_{\underline{5}}$	$\big(\chi^d_{\underline{5}}(\xi\chi)_{\underline{5}}\big)_{\underline{1}}$	$\big(\chi^d_{\underline{5}}(\varphi\chi)_{\underline{5}}\big)_{\underline{1}}\bar\eta$, $(\chi^d_{\underline{5}}\chi)_{\underline{1}}(\zeta\zeta)_{\underline{1}}$, $\Big(\chi^d_{\underline{5}}\big(\chi(\zeta\zeta)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}$, $\Big(\chi^d_{\underline{5}}\big(\chi(\zeta\zeta)_{\underline{9}}\big)_{\underline{5}}\Big)_{\underline{1}}$	$\big(\chi^d_{\underline{5}}(\xi\chi)_{\underline{5}}\big)_{\underline{1}}\eta\bar\eta$, $\Big(\chi^d_{\underline{5}}\big(\varphi(\chi\chi)_{\underline{5}}\big)_{\underline{5}}\Big)_{\underline{1}}\eta$, $\big(\chi^d_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}}\bar\eta^2$
$\zeta^d_{\underline{1}}$	0	$\zeta^d_{\underline{1}}\left(\zeta(\zeta\zeta)_{\underline{5}}\right)_{\underline{1}}$	$\zeta^d_{\underline{1}}\left(\zeta(\varphi\chi)_{\underline{5}}\right)_{\underline{1}}\eta$
$\zeta^d_{\underline{3}}$	$\big(\zeta^d_{\underline{3}}(\zeta\xi)_{\underline{3}}\big)_{\underline{1}}$	$\big(\zeta^d_{\underline{3}}(\zeta\varphi)_{\underline{3}}\big)_{\underline{1}}\bar\eta$	$\big(\zeta^d_{\underline{3}}(\zeta\xi)_{\underline{3}}\big)_{\underline{1}}\eta\bar\eta$, $(\zeta^d_{\underline{3}}\varphi)_{\underline{1}}(\chi\zeta)_{\underline{1}}\eta$, $\Big(\zeta^d_{\underline{3}}\big(\varphi(\chi\zeta)_{\underline{3}}\big)_{\underline{3}}\Big)_{\underline{1}}\eta$, $\Big(\zeta^d_{\underline{3}}\big(\varphi(\chi\zeta)_{\underline{5}}\big)_{\underline{3}}\Big)_{\underline{1}}\eta$,
$\tilde\zeta^d_{\underline{1}}$	$\tilde\zeta^d_{\underline{1}}(\zeta\zeta)_{\underline{1}}$	0	$\tilde\zeta^d_{\underline{1}}(\zeta\zeta)_{\underline{1}}\eta\bar\eta$, $\tilde\zeta^d_{\underline{1}}\left((\xi\xi)_{\underline{5}}(\varphi\chi)_{\underline{5}}\right)_{\underline{1}}$

Table 4. All terms up to quintic couplings in the flavon superpotential allowed by the flavour symmetry SO(3)$\times$U(1). μ_η and A_ξ are free parameters with one mass unit to balance the dimension in the superpotential.

JHEP11(2018)173

where ξ^{A_4} is the A_4-invariant part with each components given in eq. (4.4) and δ_ξ represents A_4-breaking corrections. Eq. (4.26) only gives an all overall small correction to v_ξ without breaking the A_4 symmetry. Eq. (4.25) is approximately simplified to

$$2(\delta_\xi \xi^{A_4})_{\underline{5}} \approx \frac{1}{\Lambda}\left(\xi^{A_4}\varphi^{Z_3}\right)_{\underline{5}}\bar{\eta} + \frac{1}{\Lambda}\chi^{Z_2}(\varphi^{Z_3}\varphi^{Z_3})_{\underline{1}} + \frac{1}{\Lambda^2}\chi^{Z_2}\eta^3 + \frac{1}{\Lambda^2}(\xi^{A_4}\xi^{A_4})_{\underline{5}}\eta\bar{\eta} + \cdots\,, \tag{4.28}$$

where $\big((\xi^{A_4}+\delta_\xi)(\xi^{A_4}+\delta_\xi)\big)_{\underline{5}} \approx 2(\delta_\xi\xi^{A_4})_{\underline{5}}$ is used on the left hand side and ξ, χ and φ are replaced by the A_4-, Z_2- and Z_3-invariant VEVs ξ^{A_4}, χ^{Z_2} and φ^{Z_3} on the right hand side of eq. (4.4), respectively. In our paper, since we only care about the order of magnitude of corrections, we neglect CG coefficients in the products and do a naive estimation of the order of magnitude. Then we obtain

$$\frac{\delta_\xi}{v_\xi} \lesssim \max\left\{\frac{v_\varphi v_{\bar{\eta}}}{\Lambda v_\xi}, \frac{v_\chi v_\varphi^2}{\Lambda v_\xi^2}, \frac{v_\chi v_\eta^3}{\Lambda^2 v_\xi^2}, 0, \cdots\right\} = \frac{v_\varphi v_{\bar{\eta}}}{\Lambda v_\xi}\,, \tag{4.29}$$

where the fourth term in the curly bracket has a vanishing contribution since $(\xi^{A_4}\xi^{A_4})_{\underline{5}} = 0$. The relation in eq. (4.7) has been used. In the above estimation, we include all corrections from table 4 and pick the largest one $v_\xi v_\varphi/(\Lambda v_\xi)$. Since $v_\chi, v_\varphi \ll v_\xi \ll \Lambda$, this correction is very small and can be safely ignored. The exact correction may be different from the estimation but must be smaller than it.

Similarly, we can estimate corrections to the VEVs of φ, χ and ζ. We denote the shifted VEVs of φ, χ and ζ as

$$\begin{aligned}
&\varphi^{Z_3} + \delta_\varphi\,,\\
&\chi^{Z_2} + \delta_\chi\,,\\
&\zeta^{\mathbf{1}'} + \delta_\zeta\,,
\end{aligned} \tag{4.30}$$

respectively, where φ^{Z_3}, χ^{Z_2} and $\zeta^{\mathbf{1}'}$ represent leading-order value in eq. (4.4) and δ_φ, δ_χ and δ_ζ are subleading order corrections. Once subleading high dimensional operators are included, the minimisation of the superpotential gives rise to

$$\begin{aligned}
&\frac{f_2}{\Lambda}\left(\delta_\xi(\varphi^{Z_3}\varphi^{Z_3})_{\underline{5}}\right)_{\underline{5}} + \frac{2f_2}{\Lambda}\left(\xi^{A_4}(\delta_\varphi\varphi^{Z_3})_{\underline{5}}\right)_{\underline{5}}\\
&\qquad\approx \frac{1}{\Lambda^2}\big((\xi^{A_4}\xi^{A_4})_{\underline{5}}\varphi^{Z_3}\big)_{\underline{5}}\eta + \frac{1}{\Lambda^2}(\varphi^{Z_3}\varphi^{Z_3})_{\underline{5}}(\zeta^{\mathbf{1}'}\zeta^{\mathbf{1}'})_{\underline{1}} + \cdots\,;\\
&\frac{g_2}{\Lambda}(\delta_\xi(\chi^{Z_2}\chi^{Z_2})_{\underline{5}})_{\underline{3}} + \frac{2g_2}{\Lambda}(\xi^{A_4}(\delta_\chi\chi^{Z_2})_{\underline{5}})_{\underline{3}} + \frac{g_3}{\Lambda}(\delta_\xi(\chi^{Z_2}\chi^{Z_2})_{\underline{9}})_{\underline{3}} + \frac{2g_3}{\Lambda}(\xi^{A_4}(\delta_\chi\chi^{Z_2})_{\underline{9}})_{\underline{3}}\\
&\qquad\approx \frac{1}{\Lambda^2}(\xi^{A_4}\chi^{Z_2})_{\underline{3}}\bar{\eta}^2 + \frac{1}{\Lambda^2}\varphi(\chi^{Z_2}\chi^{Z_2})_{\underline{1}}\bar{\eta} + \cdots\,,\\
&(\delta_\xi\chi^{Z_2})_{\underline{5}} + (\xi^{A_4}\delta_\chi)_{\underline{5}} \approx \frac{1}{\Lambda}(\varphi^{Z_3}\chi^{Z_2})_{\underline{5}}\bar{\eta} + \frac{1}{\Lambda}\chi^{Z_2}(\zeta^{\mathbf{1}'}\zeta^{\mathbf{1}'})_{\underline{1}}\cdots\,;\\
&h_2(\zeta^{\mathbf{1}'}\delta_\xi)_{\underline{3}} + h_2(\delta_\zeta\xi^{A_4})_{\underline{3}} \approx \frac{1}{\Lambda}(\zeta^{\mathbf{1}'}\varphi^{Z_3})_{\underline{3}}\bar{\eta} + \cdots\,,\\
&2h_3(\zeta^{\mathbf{1}'}\delta_\zeta)_{\underline{1}} \approx \frac{1}{\Lambda^2}(\zeta^{\mathbf{1}'}\zeta^{\mathbf{1}'})_{\underline{1}}\eta\bar{\eta} + \cdots\,.
\end{aligned} \tag{4.31}$$

A naive estimation gives the upper bounds of corrections

$$
\begin{aligned}
\frac{\delta_\varphi}{v_\varphi} &\lesssim \max\left\{\frac{\delta_\xi}{v_\xi}, 0, \frac{v_\zeta^2}{\Lambda v_\xi}, \cdots\right\} = \frac{\delta_\xi}{v_\xi} \lesssim \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}\,,\\
\frac{\delta_\chi}{v_\chi} &\lesssim \max\left\{\frac{\delta_\xi}{v_\xi}, \frac{v_{\bar\eta}^2}{\Lambda v_\chi}, \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}, \cdots\right\} = \frac{v_{\bar\eta}^2}{\Lambda v_\chi}\,,\\
\frac{\delta_\zeta}{v_\zeta} &\lesssim \max\left\{\frac{\delta_\xi}{v_\xi}, \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}, 0, \cdots\right\} = \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}\,.
\end{aligned}
\tag{4.32}
$$

Again, $(\xi^{A_4}\xi^{A_4})_{\underline{5}} = 0$, as well as $(\zeta^{\mathbf{1}'}\zeta^{\mathbf{1}'})_{\underline{1}} = 0$, and the relation in eq. (4.7) are used in the above. Upper bounds of relevant corrections to the Z_3-invariant VEV δ_φ/v_φ and the ζ VEV δ_ζ/v_ζ are as small as δ_ξ/v_ξ. The upper bound of the correction to the χ VEV is larger, $\delta_\chi/v_\chi \lesssim v_{\bar\eta}^2/(\Lambda v_\chi) \sim \sqrt{v_\eta v_{\bar\eta}}/\Lambda$. However, we calculate this correction in detail in appendix C and find that the true correction

$$
\frac{\delta_\chi}{v_\chi} \sim \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}\,,
\tag{4.33}
$$

which is also very small.

We numerically give an example of the size of these corrections. By setting

$$
A_\xi = 0.3\Lambda\,, \quad v_\eta = 0.1\Lambda\,, \quad v_{\bar\eta} = 0.03\Lambda\,,
\tag{4.34}
$$

we obtain

$$
v_\xi \sim 0.1\Lambda\,, \quad v_\varphi \sim 0.01\Lambda\,, \quad v_\chi \sim 0.03\Lambda\,, \quad v_\zeta \sim 0.001\Lambda\,,
\tag{4.35}
$$

and

$$
\frac{\delta_\xi}{v_\xi}, \frac{\delta_\varphi}{v_\varphi}, \frac{\delta_\zeta}{v_\zeta} \lesssim 0.005\,, \quad \frac{\delta_\chi}{v_\chi} \sim 0.005\,.
\tag{4.36}
$$

All corrections are less than 1%. Therefore, VEVs of ξ, χ, φ and ζ are stable under subleading corrections.

4.5 Subleading corrections to flavour mixing (are important)

At leading order, the flavour mixing appears as the tri-bimaximal pattern. Deviation arises after subleading corrections are considered. There are two origins of subleading corrections: subleading higher dimensional operators in superpotential terms for lepton mass generation w_ℓ and higher dimensional operators in the flavon superpotential w_f. The second type shift the flavon VEVs and further modify the mixing. As discussed in the last subsection, these corrections in this model are less than 1%, safely negligible. In the following, we will only discuss corrections from the first origin.

Subleading terms contributing to $\ell e^c H_d$ up to $d \leqslant 7$ and those to $\ell R_\mu H_d$ or $\ell R_\tau H_d$ up to $d \leqslant 6$ include

$$\begin{aligned}
w_{e^c} \supset\; & \frac{1}{\Lambda^4}(\varphi\ell)_{\underline{1}} e^c \eta^3 H_d + \frac{1}{\Lambda^4}\Big(\big(\xi(\varphi\varphi)_{\underline{5}}\big)_{\underline{3}}\ell\Big)_{\underline{1}} e^c \eta H_d\,, \\
w_{R_\mu} \supset\; & \frac{1}{\Lambda^2}\big((\ell R_\mu)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}} H_d + \frac{1}{\Lambda^3}\Big\{\big((\ell R_\mu)_{\underline{7}}\xi\big)_{\underline{1}}\eta\bar{\eta}H_d + \big((\ell R_\mu)_{\underline{3}}(\varphi\chi)_{\underline{3}}\big)_{\underline{1}}\eta H_d \\
& + \big((\ell R_\mu)_{\underline{5}}(\varphi\chi)_{\underline{5}}\big)_{\underline{1}}\eta H_d + \big((\ell R_\mu)_{\underline{7}}(\varphi\chi)_{\underline{7}}\big)_{\underline{1}}\eta H_d\Big\}\,, \\
w_{R_\tau} \supset\; & \frac{1}{\Lambda^2}\big((\ell R_\tau)_{\underline{7}}\xi\big)_{\underline{1}}\eta H_d + \frac{1}{\Lambda^3}\Big\{\big((\ell R_\tau)_{\underline{3}}\varphi\big)_{\underline{1}}\eta\bar{\eta}H_d + \big((\ell R_\tau)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}}\eta H_d \\
& + \Big((\ell R_\tau)_{\underline{3}}\big(\xi(\xi\xi)_{\underline{5}}\big)_{\underline{3}}\Big)_{\underline{1}} H_d + \Big((\ell R_\tau)_{\underline{3}}\big(\xi(\xi\xi)_{\underline{9}}\big)_{\underline{3}}\Big)_{\underline{1}} H_d + \big((\ell R_\tau)_{\underline{7}}\xi\big)_{\underline{1}}(\xi\xi)_{\underline{1}}H_d \\
& + \Big((\ell R_\tau)_{\underline{7}}\big(\xi(\xi\xi)_{\underline{5}}\big)_{\underline{7}}\Big)_{\underline{1}} H_d + \Big((\ell R_\tau)_{\underline{7}}\big(\xi(\xi\xi)_{\underline{9}}\big)_{\underline{7}}\Big)_{\underline{1}} H_d\Big\}\,.
\end{aligned} \tag{4.37}$$

For terms involving only some of φ, ξ, η and $\bar{\eta}$, no Z_3-breaking effects are included. The Z_3 symmetry always guarantees that the corrected effective Yukawa couplings take the forms $(1,1,1)^T$, $(1,\omega,\omega^2)^T$ and $(1,\omega^2,\omega)^T$, as in eqs. (4.10), (4.13) and (4.15), respectively. Terms breaking the Z_3 symmetry are those involving ζ or χ. There are five terms left, $\big((\ell R_\mu)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}}H_d$, $\big((\ell R_\tau)_{\underline{5}}(\zeta\zeta)_{\underline{5}}\big)_{\underline{1}}\eta H_d$, $\big((\ell R_\mu)_{\underline{3}}(\varphi\chi)_{\underline{3}}\big)_{\underline{1}}\eta H_d$, $\big((\ell R_\mu)_{\underline{5}}(\varphi\chi)_{\underline{5}}\big)_{\underline{1}}\eta H_d$, and $\big((\ell R_\mu)_{\underline{7}}(\varphi\chi)_{\underline{7}}\big)_{\underline{1}}\eta H_d$. The first two terms only contribute to coupling between ℓ and $R_{\mu\mathbf{3}}$ or $R_{\tau\mathbf{3}}$. The rest three terms contributing to couplings between ℓ and μ^c. Their contributions to the charged lepton mass matrix are characterised by adding a new matrix

$$\delta M_l = \frac{v_\eta v_\chi v_\varphi}{\Lambda^3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & c\omega + d\omega^2 & 0 \\ 0 & c\omega^2 + d\omega & 0 \end{pmatrix}\frac{v_d}{\sqrt{2}} \tag{4.38}$$

to M_l. Acting U_ω^T on the left hand side of δM_l leaves

$$U_\omega^T \delta M_l = \frac{v_\eta v_\chi v_\varphi}{\sqrt{3}\Lambda^3}\begin{pmatrix} 0 & -c-d & 0 \\ 0 & 2c-d & 0 \\ 0 & 2c-d & 0 \end{pmatrix}\frac{v_d}{\sqrt{2}}\,, \tag{4.39}$$

where c and d are real dimensionless parameters. The unitary matrix to diagonalise M_l is modified to $U_l \simeq U_\omega U_{e\mu}$, where $U_{e\mu}$ is a complex rotation matrix on the $e\mu$ plane,

$$U_{e\mu} = \begin{pmatrix} \cos\theta_{e\mu} & \sin\theta_{e\mu}e^{-i\phi_{e\mu}} & 0 \\ -\sin\theta_{e\mu}e^{i\phi_{e\mu}} & \cos\theta_{e\mu} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.40}$$

with

$$\begin{aligned}
\sin\theta_{e\mu} &= \frac{(c+d)v_\eta v_\chi}{y_\mu v_{\bar{\eta}}\Lambda}\,, \\
\phi_{e\mu} &= \arg\big(-(c+d)v_\eta v_\chi y_\mu v_{\bar{\eta}}\Lambda\big)\,.
\end{aligned} \tag{4.41}$$

Here, we have ignored the $(3,2)$ entry of $U_\omega^T \delta M_l$ since it is too small compared with the τ mass $m_\tau \sim v_\varphi v_d/\Lambda$.

In the neutrino sector, terms for neutrino masses up to $d \leqslant 5$ have only trivial corrections, $w_N \supset \frac{1}{\Lambda^2}\{(\ell N)_{\underline{1}}\eta\bar{\eta}H_u + \left(\chi(NN)_{\underline{5}}\right)_{\underline{1}}\eta\bar{\eta}\}$. Therefore, the unitary matrix U_ν to diagonal M_ν keeps the same as that in the leading order.

Including the subleading correction, the PMNS matrix is modified into $U_{\rm PMNS} = U_{e\mu}^\dagger U_{\rm TBM}$. multiplying $U_{e\mu}$ on the left hand side does not change the third row of the PMNS matrix. Three mixing angles are given by [6–8]

$$\begin{aligned}
\sin\theta_{13} &= \frac{\sin\theta_{e\mu}}{\sqrt{2}}\,,\\
\sin\theta_{12} &= \sqrt{\frac{2-2\sin 2\theta_{e\mu}\cos\phi_{e\mu}}{3(2-\sin^2\theta_{e\mu})}}\,,\\
\sin\theta_{23} &= \frac{\cos\theta_{e\mu}}{\sqrt{2-\sin^2\theta_{e\mu}}}\,.
\end{aligned} \tag{4.42}$$

In this model, θ_{23} in the first octant is predicted. The reactor angle $\theta_{13} \sim v_\eta v_\chi/(v_{\bar\eta}\Lambda)$. For the numerical value in eq. (4.35), we have $v_\eta v_\chi/(v_{\bar\eta}\Lambda) \sim 0.05$. In order to generate sizeable value of θ_{13}, a relatively large value of the ratio $(c+d)/y_\mu$ is required. This is not hard to be achieved. The Dirac-type CP-violating phase is predicted to be

$$\delta = \arg\left((3\cos 2\theta_{e\mu} + \cos 4\theta_{e\mu})\cos\phi_{e\mu} - i(\cos 2\theta_{e\mu}+3)\sin\phi_{e\mu} + \sin 2\theta_{e\mu}\right). \tag{4.43}$$

The unknown phase $\phi_{e\mu}$ can be eliminated to yield sum rules which have been widely studied [6–8, 59–64]. In the limit $\phi_{e\mu} \to \pi/2$, an almost maximal CP-violating phase $\delta \sim 3\pi/2$ is predicted.

4.6 Phenomenological implications of gauged SO(3)

We label the gauge field of SO(3) and U(1) as $F'^{\,1,2,3}$ and B', respectively. Their interactions with flavons or fermions are simply obtained with the replacement

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{g}_3' \sum_{a=1,2,3} F'^{\,a}_\mu \tau^a + Q\mathrm{g}_1' B'_\mu\,, \tag{4.44}$$

in the kinetic terms of the relevant fields. Here, g_3' and g_1' are gauge couplings of SO(3) and U(1), respectively, and the U(1) charge Q for each field is listed in table 3.

Specifically, the kinetic term for ξ in eq. (2.43) is replaced by $(D_\mu\xi^* D^\mu\xi)_{\underline{1}}$ with

$$(D_\mu\xi)_{ijk} = (\partial_\mu\xi)_{ijk} + \mathrm{g}_3' \sum_{a=1,2,3} F'^{\,a}_\mu \left[(\tau^a)_{il}\xi_{ljk} + (\tau^a)_{jl}\xi_{ilk} + (\tau^a)_{kl}\xi_{ijl}\right] + Q\mathrm{g}_1' B'_\mu \xi_{ijk}\,. \tag{4.45}$$

where $Q = +1$ for ξ has been used. $F'^{\,1,2,3}_\mu$ gain masses once ξ get the A_4-invariant VEV. We obtain that $M^2_{F'^1} = M^2_{F'^2} = M^2_{F'^3} = (2\mathrm{g}_3' v_\xi)^2$. The degenerate mass spectrum is also consistent with the A_4 symmetry.[4] Later after the rest flavons ζ, φ and χ gain VEVs, mass

[4] One may use the generators s and t to perform a A_4 transformation, an A_4-invariant mass term for F' is obtained only if all masses are degenerated.

splitting are generated among $F'^{1,2,3}$. Since VEVs of ζ, φ and χ are much smaller than that of ξ, the mass splittings are very small, and masses of $F'^{1,2,3}$ are still nearly degenerate.

B' obtains a mass from VEVs of both ξ and η, $\bar{\eta}$, $M_{B'}^2 = g_1'^2(v_\xi^2 + v_\eta^2 + v_{\bar{\eta}}^2)$. After A_4 breaking, VEVs of ζ, φ and χ contribute small corrections to the B' mass. Interactions between leptons and B' are flavour-dependent, with charges for ℓ, e^c, μ^c and τ^c given by $-\frac{2}{3}$, $-\frac{7}{3}$, -1 and $-\frac{1}{3}$, respectively.

In the limit of the A_4 invariance, there is no mixing between F' and B'. This can be simply explained as follows. The mixing between F' and B' from eq. (4.45) and $(D_\mu \xi^* D^\mu \xi)_{\underline{1}}$, if exists, can be only generated via coupling $F'B'\xi\xi$. Since $F' \sim \underline{3}$, $B' \sim \underline{1}$, $\xi \sim \underline{7}$, the only SO(3) invariant formed by these fields is $B_\mu\big(F^\mu(\xi^*\xi)_{\underline{3}}\big)_{\underline{1}}$. Here, the $\underline{3}$-plet contraction between ξ^* and ξ are anti-symmetric. Once ξ get the VEV, where only one of the seven components has a non-zero value, $\langle \xi_0 \rangle = v_\xi$, the anti-symmetric contraction $\langle (\xi^*\xi)_{\underline{3}} \rangle$ vanishes. Therefore, there is no mixing between F' and B'. The mixing between F' and B' is generated after A_4 breaking, induced by terms such as $B_\mu\big(F^\mu(\xi^*\chi)_{\underline{3}}\big)_{\underline{1}}$. The resulted mixing between F' and B' is suppressed by the ratio v_χ/v_ξ.

These gauge bosons are supposed to be very heavy, with masses around $\sim \mathcal{O}(v_\xi)$ or $\sim \mathcal{O}(\max(v_\xi, v_\eta, v_{\bar{\eta}}))$, respectively, if gauge coefficients are of order one. However, they could be much lighter if gauge couplings are tiny. For example, if Λ is fixed at 10^4 TeV, v_ξ and v_η are predicted to be around 10^3 TeV and v_χ, v_φ and v_ζ be around 100 TeV. For a gauge coupling around 10^{-3}, TeV-scale gauge bosons are predicted. Then, interesting signatures involving gauge interactions can be tested at colliders or precision measurements of charged leptons. Another interesting point is the prediction of a heavy tau lepton with mass also around TeV scale ($v_\zeta^2/\Lambda \sim 1$ TeV). Its interaction with B' can be tested at colliders.

4.7 Absence of domain walls

The domain wall problem is a well-known problem for discrete symmetry breaking. In this paper, all flavour symmetries at high scale are gauged. A_4, and the residual symmetries Z_3 and Z_2, are just phenomenologically effective symmetries at lower scales. The usual domain wall problem for the global symmetry breaking does not apply here.

In our model, we actually have a two-step phase transition SO(3) $\to A_4$ and $A_4 \to Z_3, Z_2$. We discuss more on why the topological defect of domain walls does not exit in the model.

At the first step, SO(3) $\to A_4$, the breaking of a gauge symmetry does not introduce domain walls. As noted in section 2, there are degenerate vacuums which are continuously connected by SO(3) basis transformation as in eq. (2.1). All vacuums are perturbatively equivalent.

At the second step, $A_4 \to Z_3, Z_2$, degenerate Z_3-invariant or Z_2-invariant vacuums exit, as shown in eqs. (3.3) and (3.6). Taking the Z_3-invariant vacuum as an example, different Z_3-invariant vacuums are randomly generated during A_4 breaking to Z_3 and domain walls separating different vacuums arise. These domain walls store energy with energy density inside the wall around v_φ^4 or v_χ^4. Without considering gauge interactions, there are not enough energy inputted to force one vacuum jumping across the wall into another. Therefore, domain walls survive. Once gauge interactions are included, domain walls should decay to light particles mediated by gauge bosons. For the case of small gauge couplings, the gauge bosons may be light enough, i.e., $M_{F'} \lesssim v_\varphi$, and domain walls may directly decay into gauge bosons.

5 Conclusion

In this paper we have discussed the breaking of SO(3) down to finite family symmetries such as A_4, S_4 and A_5 using *supersymmetric* potentials for the first time. We have analysed in detail the case of supersymmetric A_4 and its finite subgroups Z_3 and Z_2. We have proposed a supersymmetric A_4 model of leptons along these lines, originating from SO(3) $\times$ U(1), which leads to a phenomenologically acceptable pattern of lepton mixing and masses once subleading corrections are taken into account. We have also discussed the phenomenological consequences of having a gauged SO(3), leading to massive gauge bosons, and have shown that all domain wall problems are resolved in this model.

The main achievement of the paper is to show for the first time that *supersymmetric* SO(3) flavour symmetry can be the origin of finite non-Abelian family symmetry models. By focussing in detail on a supersymmetric A_4 model, we have demonstrated that such a strategy can lead to a viable lepton model which can explain all oscillation data with SUSY being preserved in the low energy spectrum (below the flavour symmetry breaking scales). Moreover, we have shown that, if the SO(3) is gauged, there may be interesting phenomenological implications due to the massive gauge bosons.

About a half of the paper is devoted to the study of the realistic supersymmetric A_4 model of leptons, arising from SO(3) $\times$ U(1). This study is important in order to verify that it is really possible to construct a fully working model along these lines. The main achievements of the specific model may be summarised as follows:

- We have achieved the breaking of SO(3) $\to A_4$ in SUSY, using high irreps of SO(3) and flat directions. In this paper, we have chosen a $\underline{7}$-plet, i.e., a rank-3 tensor in 3d space, to achieve the breaking. We have shown that it is possible to break SO(3) to S_4 or A_5 by using different higher irreps.

- We have shown that it is possible to also achieve, at the level of SO(3), the subsequent breaking of A_4 at a lower scale (below the SO(3) breaking scale) to the residual symmetries Z_3 and Z_2. Such Z_3 and Z_2 symmetries are preserved in charged lepton sector and neutrino sector, respectively, after the A_4 breaking, in accordance with the semi-direct model building strategy.

- Starting from a supersymmetric flavour group SO(3) $\times$ U(1), we have shown how SO(3) is broken first to A_4, and then to Z_3 and Z_2. The A_4, Z_3 and Z_2 symmetries are respectively achieved by the flavons ξ, φ and χ after they gain the A_4-, Z_3- and Z_2-invariant VEVs, respectively. We have found that tri-bimaximal mixing (with zero reactor angle) is realised at leading order. One technical point is that the singlet irreps $\mathbf{1}'$ and $\mathbf{1}''$ of A_4 always accompany each other after SO(3) breaking. To avoid any fine tuning of parameters related to μ and τ masses, we have introduced an additional flavon ζ to split the $\mathbf{1}'$ and $\mathbf{1}''$.

- We have considered the influence of the higher dimensional operator corrections to the model. We have shown that the A_4-, Z_3- and Z_2-invariant VEVs are stable even after subleading corrections are included. However, we have seen that the charged lepton mass matrix is modified by higher dimensional operators, due to the coupling with χ, which gains the Z_2-invariant VEV. This welcome correction leads to additional mixing between e and μ, giving rise to a non-zero θ_{13} and the CP-violating phase δ.
- If the SO(3) $\times$ U(1) is gauged, the model predicts three gauge bosons $F'^{1,2,3}$ with the nearly degenerate masses after SO(3) breaking to A_4. Another gauge boson B' gain a mass after U(1) is broken. These gauge bosons with their flavour-dependent interactions with leptons will lead to phenomenological signatures worthy of further study.
- We emphasise that the flavour symmetry at high scale is the continuous gauge symmetry SO(3) $\times$ U(1), with no *ad hoc* discrete symmetries introduced, and A_4 being just an effective flavour symmetry below the SO(3) breaking scale. We have shown that the usual domain wall problems encountered in A_4 models are resolved here.

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A Clebsch-Gordan coefficients of SO(3)

In SO(3), the product of two irreducible representations (irreps) ϕ of dimension $2p+1$ and Ψ of dimension $2q+1$ are decomposed as follows:

$$(\underline{2p{+}1}) \times (\underline{2q{+}1}) = (\underline{2|p{-}q|{+}1}) + (\underline{2|p{-}q|{+}3}) + \cdots + (\underline{2(p{+}q){+}1}) \tag{A.1}$$

Some useful Clebsch-Gordan coefficients of these products in the 3d space are listed in the following:

- For $\phi \sim \Psi \sim \underline{3}$, $\underline{3} \times \underline{3} = \underline{1} + \underline{3} + \underline{5}$,

$$\begin{aligned} (\phi\Psi)_{\underline{1}} &\sim \phi_a \Psi_a \,, \\ \big((\phi\Psi)_{\underline{3}}\big)_i &\sim \epsilon_{iab}\phi_a\Psi_b \,, \\ \big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \phi_{ia}\Psi_{ja} - \frac{1}{3}\delta_{ij}\phi_a\Psi_a + (\text{perms of } ij) \,. \end{aligned} \tag{A.2}$$

- For $\phi \sim \underline{3}$ and $\Psi \sim \underline{5}$, $\underline{3} \times \underline{5} = \underline{3} + \underline{5} + \underline{7}$,

$$\begin{aligned}
\big((\phi\Psi)_{\underline{3}}\big)_i &\sim \phi_a \Psi_{ia}\,, \\
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \epsilon_{iab}\phi_a\Psi_{jb} + (\text{perms of } ij)\,, \\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \phi_i\Psi_{jk} - \frac{2}{5}\delta_{ij}\phi_a\Psi_{ka} + (\text{perms of } ijk)\,.
\end{aligned} \tag{A.3}$$

- For $\phi \sim \underline{3}$, $\Psi \sim \underline{7}$, $\underline{3} \times \underline{7} = \underline{5} + \underline{7} + \underline{9}$,

$$\begin{aligned}
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \phi_a \Psi_{ija} + (\text{perms of } ij)\,, \\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \epsilon_{iab}\phi_a\Psi_{jkb} + (\text{perms of } ijk)\,, \\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \phi_i\Psi_{jkl} - \frac{3}{7}\delta_{ij}\phi_a\Psi_{kla} + (\text{perms of } ijkl)\,.
\end{aligned} \tag{A.4}$$

- For $\phi \sim \underline{3}$, $\Psi \sim \underline{9}$, $\underline{3} \times \underline{9} = \underline{7} + \underline{9} + \underline{11}$,

$$\begin{aligned}
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \phi_a \Psi_{ijka} + (\text{perms of } ijk)\,, \\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \epsilon_{iab}\phi_a\Psi_{jklb} + (\text{perms of } ijkl)\,, \\
\big((\phi\Psi)_{\underline{11}}\big)_{ijklm} &\sim \phi_i\Psi_{jklm} - \frac{4}{9}\delta_{ij}\phi_a\Psi_{klma} + (\text{perms of } ijklm)\,.
\end{aligned} \tag{A.5}$$

- For $\phi \sim \Psi \sim \underline{5}$, $\underline{5} \times \underline{5} = \underline{1} + \underline{3} + \underline{5} + \underline{7} + \underline{9}$,

$$\begin{aligned}
(\phi\Psi)_{\underline{1}} &\sim \phi_{ab}\Psi_{ab}\,, \\
\big((\phi\Psi)_{\underline{3}}\big)_i &\sim \epsilon_{iab}\phi_{ac}\Psi_{bc}\,, \\
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \phi_{ia}\Psi_{ja} - \frac{1}{3}\delta_{ij}\phi_{ab}\Psi_{ab} + (\text{perms of } ij)\,, \\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \epsilon_{iab}\phi_{ja}\Psi_{kb} - \frac{1}{5}\epsilon_{iab}\delta_{jk}\phi_{ac}\Psi_{bc} + (\text{perms of } ijk)\,, \\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \phi_{ij}\Psi_{kl} - \frac{4}{7}\delta_{ij}\phi_{ka}\Psi_{la} + \frac{2}{35}\delta_{ij}\delta_{kl}\phi_{ab}\Psi_{ab} + (\text{perms of } ijkl)\,.
\end{aligned} \tag{A.6}$$

- For $\phi \sim \underline{5}$, $\Psi \sim \underline{7}$, $\underline{5} \times \underline{7} = \underline{3} + \underline{5} + \underline{7} + \underline{9} + \underline{11}$,

$$\begin{aligned}
\big((\phi\Psi)_{\underline{3}}\big)_i &\sim \phi_{ab}\Psi_{iab}\,, \\
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \epsilon_{iab}\phi_{ac}\Psi_{jbc} + (\text{perms of } ij)\,, \\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \phi_{ia}\Psi_{jka} - \frac{2}{5}\delta_{ij}\phi_{ab}\Psi_{kab} + (\text{perms of } ijk)\,, \\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \epsilon_{iab}\phi_{ja}\Psi_{klb} - \frac{2}{7}\epsilon_{iab}\delta_{jk}\phi_{ac}\Psi_{lbc} + (\text{perms of } ijkl)\,, \\
\big((\phi\Psi)_{\underline{11}}\big)_{ijkl} &\sim \phi_{ij}\Psi_{klm} - \frac{2}{3}\delta_{ij}\phi_{ka}\Psi_{lma} + \frac{2}{21}\delta_{ij}\delta_{kl}\phi_{ab}\Psi_{mab} + (\text{perms of } ijkl)\,.
\end{aligned} \tag{A.7}$$

- For $\phi \sim \Psi \sim \underline{7}$, $\underline{7}\times\underline{7} = \underline{1}+\underline{3}+\underline{5}+\underline{7}+\underline{9}+\underline{11}+\underline{13}$,

$$
\begin{aligned}
(\phi\Psi)_{\underline{1}} &\sim \phi_{abc}\Psi_{abc}\,,\\
\big((\phi\Psi)_{\underline{3}}\big)_i &\sim \epsilon_{iab}\phi_{acd}\Psi_{bcd}\,,\\
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \phi_{iab}\Psi_{jab} - \frac{1}{3}\delta_{ij}\phi_{abc}\Psi_{abc} + (\text{perms of } ij)\,,\\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \epsilon_{iab}\phi_{jac}\Psi_{kbc} - \frac{1}{5}\epsilon_{iab}\delta_{jk}\phi_{acd}\Psi_{bcd} + (\text{perms of } ijk)\,,\\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \phi_{ija}\Psi_{kla} - \frac{4}{7}\delta_{ij}\phi_{kab}\Psi_{lab} + \frac{2}{35}\delta_{ij}\delta_{kl}\phi_{abc}\Psi_{abc} + (\text{perms of } ijkl)\,,\\
\big((\phi\Psi)_{\underline{11}}\big)_{ijklm} &\sim \epsilon_{iab}\phi_{jka}\Psi_{lmb} - \frac{4}{9}\epsilon_{iab}\delta_{jk}\phi_{lac}\Psi_{mbc} + \frac{2}{63}\epsilon_{iab}\delta_{jk}\delta_{lm}\phi_{acd}\Psi_{bcd}\\
&\quad + (\text{perms of } ijklm)\,,\\
\big((\phi\Psi)_{\underline{13}}\big)_{ijklmn} &\sim \phi_{ijk}\Psi_{lmn} - \frac{9}{11}\delta_{ij}\phi_{kla}\Psi_{mna} + \frac{2}{11}\delta_{ij}\delta_{kl}\phi_{mab}\Psi_{nab}\\
&\quad - \frac{2}{231}\delta_{ij}\delta_{kl}\delta_{mn}\phi_{abc}\Psi_{abc} + (\text{perms of } ijklmn)\,.
\end{aligned}
\tag{A.8}
$$

- For $\phi \sim \underline{7}$, $\Psi \sim \underline{9}$, $\underline{7}\times\underline{9} = \underline{3}+\underline{5}+\underline{7}+\underline{9}+\underline{11}+\underline{13}+\underline{15}$,

$$
\begin{aligned}
\big((\phi\Psi)_{\underline{3}}\big)_i &\sim \phi_{abc}\Psi_{iabc}\,,\\
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \epsilon_{iab}\phi_{acd}\Psi_{jbcd} + (\text{perms of } ij)\,,\\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \phi_{iab}\Psi_{jkab} - \frac{2}{5}\delta_{ij}\phi_{abc}\Psi_{kabc} + (\text{perms of } ijk)\,,\\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \epsilon_{iab}\phi_{jac}\Psi_{klbc} - \frac{2}{7}\epsilon_{iab}\delta_{jk}\phi_{acd}\Psi_{lbcd} + (\text{perms of } ijkl)\,,\\
\big((\phi\Psi)_{\underline{11}}\big)_{ijkl} &\sim \phi_{ija}\Psi_{klma} - \frac{2}{3}\delta_{ij}\phi_{kab}\Psi_{lmab} + \frac{2}{21}\delta_{ij}\delta_{kl}\phi_{abc}\Psi_{mabc}\\
&\quad + (\text{perms of } ijkl)\,.
\end{aligned}
\tag{A.9}
$$

- For $\phi \sim \Psi \sim \underline{9}$, $\underline{9}\times\underline{9} = \underline{1}+\underline{3}+\underline{5}+\underline{7}+\underline{9}+\underline{11}+\underline{13}+\underline{15}+\underline{17}$,

$$
\begin{aligned}
(\phi\Psi)_{\underline{1}} &\sim \phi_{abcd}\Psi_{abcd}\,,\\
\big((\phi\Psi)_{\underline{3}}\big)_i &\sim \epsilon_{iab}\phi_{acdf}\Psi_{bcdf}\,,\\
\big((\phi\Psi)_{\underline{5}}\big)_{ij} &\sim \phi_{iabc}\Psi_{jabc} - \frac{1}{3}\delta_{ij}\phi_{abcd}\Psi_{abcd} + (\text{perms of } ij)\,,\\
\big((\phi\Psi)_{\underline{7}}\big)_{ijk} &\sim \epsilon_{iab}\phi_{jacd}\Psi_{kbcd} - \frac{1}{5}\epsilon_{iab}\delta_{jk}\phi_{acdf}\Psi_{bcdf} + (\text{perms of } ijk)\,,\\
\big((\phi\Psi)_{\underline{9}}\big)_{ijkl} &\sim \phi_{ijab}\Psi_{klab} - \frac{4}{7}\delta_{ij}\phi_{kabc}\Psi_{labc} + \frac{2}{35}\delta_{ij}\delta_{kl}\phi_{abcd}\Psi_{abcd}\\
&\quad + (\text{perms of } ijkl)\,,\\
\big((\phi\Psi)_{\underline{11}}\big)_{ijklm} &\sim \epsilon_{iab}\phi_{jkac}\Psi_{lmbc} - \frac{4}{9}\epsilon_{iab}\delta_{jk}\phi_{lacd}\Psi_{mbcd}\\
&\quad + \frac{2}{63}\epsilon_{iab}\delta_{jk}\delta_{lm}\phi_{acdf}\Psi_{bcdf} + (\text{perms of } ijklm)\,,\\
\big((\phi\Psi)_{\underline{13}}\big)_{ijklmn} &\sim \phi_{ijka}\Psi_{lmna} - \frac{9}{11}\delta_{ij}\phi_{klab}\Psi_{mnab} + \frac{2}{11}\delta_{ij}\delta_{kl}\phi_{mabc}\Psi_{nabc}\\
&\quad - \frac{2}{231}\delta_{ij}\delta_{kl}\delta_{mn}\phi_{abcd}\Psi_{abcd} + (\text{perms of } ijklmn)\,.
\end{aligned}
\tag{A.10}
$$

- For $\phi \sim \Psi \sim \underline{13}$, $\underline{13} \times \underline{13} = \underline{1} + \underline{3} + \underline{5} + \underline{7} + \underline{9} + \underline{11} + \underline{13} + \underline{15} + \underline{17} + \underline{19} + \underline{21} + \underline{23} + \underline{25}$,

$$\begin{aligned}
(\phi\Psi)_{\underline{1}} &\sim \phi_{abcdfg}\Psi_{abcdfg}\,,\\
\left((\phi\Psi)_{\underline{5}}\right)_{ij} &\sim \phi_{iabcdf}\Psi_{jabcdf} - \frac{1}{3}\delta_{ij}\phi_{abcdfg}\Psi_{abcdfg} + (\text{perms of } ij)\,,\\
\left((\phi\Psi)_{\underline{9}}\right)_{ijkl} &\sim \phi_{ijabcd}\Psi_{klabcd} - \frac{4}{7}\delta_{ij}\phi_{kabcdf}\Psi_{labcdf} + \frac{2}{35}\delta_{ij}\delta_{kl}\phi_{abcdfg}\Psi_{abcdfg}\\
&\quad + (\text{perms of } ijkl)\,.
\end{aligned} \tag{A.11}$$

B Solutions of the superpotential minimisation

B.1 Solutions for SO(3) → A_4

Equations for the minimisation of the superpotential term w_ξ in eqs. (2.12) and (2.13) are respectively and explicitly written out as

$$\begin{aligned}
-\frac{\mu_\xi^2}{c_1} + 2\xi_{111}^2 + 3\xi_{111}\xi_{133} + 2\xi_{112}^2 + \xi_{112}\xi_{233} + 3\xi_{113}^2 + 3\xi_{113}\xi_{333}&\\
+3\xi_{123}^2 + 3\xi_{133}^2 + 2\xi_{233}^2 + 2\xi_{333}^2 &= 0\,;
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
2\left(\xi_{111}^2 + \xi_{112}^2 - \xi_{112}\xi_{233} - 3\xi_{113}\xi_{333} - 2\xi_{233}^2 - 2\xi_{333}^2\right) &= 0\,,\\
3\xi_{111}\xi_{233} - 3\xi_{112}\xi_{133} - 6\xi_{123}\xi_{333} + 6\xi_{133}\xi_{233} &= 0\,,\\
3\xi_{111}(2\xi_{113} + \xi_{333}) + 6\xi_{112}\xi_{123} + 9\xi_{113}\xi_{133} + 6\xi_{123}\xi_{233} + 6\xi_{133}\xi_{333} &= 0\,,\\
-6\xi_{111}\xi_{123} + 6\xi_{112}\xi_{113} + 3\xi_{112}\xi_{333} - 3\xi_{113}\xi_{233} &= 0\,,\\
2\left(-2\xi_{111}^2 - 3\xi_{111}\xi_{133} - 2\xi_{112}^2 - \xi_{112}\xi_{233} + \xi_{233}^2 + \xi_{333}^2\right) &= 0\,.
\end{aligned} \tag{B.2}$$

Five equations in eq. (B.2) corresponds to it (11), (12), (13), (23) and (33) entries of two rank-2 tensor $(\xi\xi)_{\underline{5}} \equiv \partial w_\xi/(c_2\partial\xi^d_{\underline{5}})$, respectively. By setting $\xi_{111} = \xi_{112} = \xi_{113} = \xi_{133} = \xi_{233} = \xi_{333} = 0$, eq. (B.2) is automatically satisfied. Then, eq. (B.1) is left with

$$-\frac{\mu_\xi^2}{c_1} + 3\xi_{123}^2 = 0\,, \tag{B.3}$$

from which we obtain $\xi_{123} = \pm\sqrt{\mu_\xi^2/(3c_1)}$. Then, we arrive at the special solution in eq. (2.9).

B.2 Solutions for A_4 → Z_3

Equations for the minimasation of w_φ is given in eq. (3.2). Taking the VEV of ξ in eq. (2.9) into these equations, i.e., $\xi_{111} = \xi_{112} = \xi_{113} = \xi_{133} = \xi_{233} = \xi_{333} = 0$, part of these equations are automatically satisfied, the left vanishing ones are simplified as

$$\begin{aligned}
\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \frac{\mu_\varphi^2}{f_1} &= 0\,,\\
4\xi_{123}\left(\varphi_2^2 - \varphi_3^2\right) &= 0\,,\\
-4\xi_{123}\left(\varphi_2^2 - \varphi_1^2\right) &= 0\,.
\end{aligned} \tag{B.4}$$

It is straightforward to derive all solutions in eq. (3.3).

B.3 Solutions for $A_4 \to Z_2$

Equations of minimisation of w_χ are given in eq. (3.5). After ξ get the A_4-invariant VEV, they are explicitly written out as

$$\chi_{11}^2 + \chi_{11}\chi_{33} + \chi_{33}^2 + \chi_{12}^2 + \chi_{13}^2 + \chi_{23}^2 - \frac{\mu_\chi^2}{2g_1} = 0\,; \tag{B.5}$$

$$\begin{aligned} v_\xi \chi_{11}\chi_{23}\left(\tfrac{72\sqrt{6}}{7}g_3 - 2\sqrt{\tfrac{2}{3}}g_2\right) + v_\xi \chi_{12}\chi_{13}\left(2\sqrt{\tfrac{2}{3}}g_2 + \tfrac{96\sqrt{6}}{7}g_3\right) &= 0\,,\\ v_\xi(\chi_{11}+\chi_{33})\chi_{13}\left(2\sqrt{\tfrac{2}{3}}g_2 - \tfrac{72\sqrt{6}}{7}g_3\right) + v_\xi \chi_{12}\chi_{23}\left(2\sqrt{\tfrac{2}{3}}g_2 + \tfrac{96\sqrt{6}}{7}g_3\right) &= 0\,,\\ v_\xi \chi_{12}\chi_{33}\left(\tfrac{72\sqrt{6}}{7}g_3 - 2\sqrt{\tfrac{2}{3}}g_2\right) + v_\xi \chi_{13}\chi_{23}\left(2\sqrt{\tfrac{2}{3}}g_2 + \tfrac{96\sqrt{6}}{7}g_3\right) &= 0\,; \end{aligned} \tag{B.6}$$

$$\begin{aligned} -g_4\sqrt{\tfrac{2}{3}}v_\xi(\chi_{11}+2\chi_{33}) &= 0\,,\\ g_4\sqrt{\tfrac{2}{3}}v_\xi(2\chi_{11}+\chi_{33}) &= 0\,. \end{aligned} \tag{B.7}$$

Eq. (B.7) leads to $\chi_{11} = \chi_{33} = 0$. Taking it to eq. (B.6), we are left with $\chi_{12}\chi_{13} = \chi_{12}\chi_{23} = \chi_{13}\chi_{23} = 0$, and therefore two of $\chi_{12}, \chi_{13}, \chi_{23}$ vanishing. The only non-vanishing one is determined by eq. (B.5). All solutions are listed here,

$$\begin{pmatrix} \langle\chi_{11}\rangle \\ \langle\chi_{12}\rangle \\ \langle\chi_{13}\rangle \\ \langle\chi_{23}\rangle \\ \langle\chi_{33}\rangle \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \pm\frac{v_\chi}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pm\frac{v_\chi}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm\frac{v_\chi}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}. \tag{B.8}$$

Representing χ_{ij} by χ', χ'' and $\chi_{1,2,3}$ in eq. (2.38), we obtain the result in eq. (3.6).

C Deviation from the Z_2-invariant vacuum

The naive estimation only gives the upper bound of the correction. The true correction may be smaller than it. It happens for the correction to the VEV of χ. The minimisation of the superpotential including subleading higher dimensional operators is given by

$$\begin{aligned} &\frac{g_2}{\Lambda}(\delta_\xi(\chi^{Z_2}\chi^{Z_2})_{\underline{5}})_{\underline{3}} + \frac{2g_2}{\Lambda}(\xi^{A_4}(\delta_\chi\chi^{Z_2})_{\underline{5}})_{\underline{3}} + \frac{g_3}{\Lambda}(\delta_\xi(\chi^{Z_2}\chi^{Z_2})_{\underline{9}})_{\underline{3}} + \frac{2g_3}{\Lambda}(\xi^{A_4}(\delta_\chi\chi^{Z_2})_{\underline{9}})_{\underline{3}} \\ &\qquad \approx \frac{1}{\Lambda^2}(\xi^{A_4}\chi^{Z_2})_{\underline{3}}\bar{\eta}^2 + \frac{1}{\Lambda^2}\varphi(\chi^{Z_2}\chi^{Z_2})_{\underline{1}}\bar{\eta} + \cdots\,, \\ &(\delta_\xi\chi^{Z_2})_{\underline{5}} + (\xi^{A_4}\delta_\chi)_{\underline{5}} \approx \frac{1}{\Lambda}(\varphi^{Z_3}\chi^{Z_2})_{\underline{5}}\bar{\eta} + \frac{1}{\Lambda}\chi^{Z_2}(\zeta^{\mathbf{1}'}\zeta^{\mathbf{1}'})_{\underline{1}}\cdots\,. \end{aligned} \tag{C.1}$$

Ignoring all the other subleading operators, we calculate its correction in detail instead of using the naive estimation. In this case, eq. (C.1) is simplified to

$$\begin{aligned} \frac{2g_2}{\Lambda}(\xi^{A_4}(\delta_\chi\chi^{Z_2})_{\underline{5}})_{\underline{3}} + \frac{2g_3}{\Lambda}(\xi^{A_4}(\delta_\chi\chi^{Z_2})_{\underline{9}})_{\underline{3}} &\approx \frac{1}{\Lambda^2}(\xi^{A_4}\chi^{Z_2})_{\underline{3}}\bar{\eta}^2\,,\\ (\xi^{A_4}\delta_\chi)_{\underline{5}} &\approx 0\,. \end{aligned} \tag{C.2}$$

Here, we has ignored the correction to the ξ VEV since it is too small as discussed in the above. The above equation is explicitly written out as

$$\begin{pmatrix} \left(\frac{72}{7}g_3 - \frac{2}{3}g_2\right)(\delta_{\chi'} + \delta_{\chi''}) \\ \sqrt{\frac{2}{3}}\left(g_2 + \frac{144}{7}g_3\right)\delta_{\chi_3} \\ \sqrt{\frac{2}{3}}\left(g_2 + \frac{144}{7}g_3\right)\delta_{\chi_2} \end{pmatrix} \frac{v_\xi v_\chi}{\Lambda} \approx \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{v_\xi v_\chi v_{\bar\eta}^2}{\sqrt{3}\Lambda^2}\,,$$
$$\begin{pmatrix} \delta_{\chi''} - \delta_{\chi'} & 0 & 0 \\ 0 & \omega\delta_{\chi''} - \omega^2\delta_{\chi'} & 0 \\ 0 & 0 & \omega^2\delta_{\chi''} - \omega\delta_{\chi'} \end{pmatrix} i\sqrt{\frac{2}{3}}v_\xi \approx 0\,. \tag{C.3}$$

This equation cannot give a self-consistent solution for $\delta_{\chi'}$ or $\delta_{\chi''}$ since the first equation predict $(\delta_{\chi'} + \delta_{\chi''})/v_\chi \sim v_{\bar\eta}^2/(\Lambda v_\chi)$ and the second one gives $\delta_{\chi'}/v_\chi \sim \delta_{\chi''}/v_\chi \sim 0$. It means that after subleading higher dimensional operators are included in the flavon superpotential, $\partial w_f/\partial\chi^d_{\underline{3}} = 0$ and $\partial w_f/\partial\chi^d_{\underline{5}} = 0$ cannot hold at the same time. In other word, there is no flat direction for the flavon.

Without flat direction, one has to calculate the VEV correction via the minimisation of the flavon potential. For similar discussion in only non-Abelian discrete symmetry, see e.g., ref. [65]. In the model discussed here, the flavon potential is given by

$$V_f = \left|\frac{\partial w_f}{\partial\chi^d_{\underline{3}}}\right|^2 + \left|\frac{\partial w_f}{\partial\chi^d_{\underline{5}}}\right|^2 + \cdots\,. \tag{C.4}$$

Taking the superpotential terms in table 4 to V_f, we see that the first term is much smaller than the second term, $\left|\frac{\partial w_f}{\partial\chi^d_{\underline{3}}}\right|^2 \ll \left|\frac{\partial w_f}{\partial\chi^d_{\underline{5}}}\right|^2$. Therefore, the minimisation of V_f is approximate to $\partial w_f/\partial\chi^d_{\underline{5}} = 0$, and the correction is given by

$$\frac{\delta_\chi}{v_\chi} \sim \max\left\{\frac{\delta_\xi}{v_\xi}, \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}, \cdots\right\} = \frac{v_\varphi v_{\bar\eta}}{\Lambda v_\xi}\,. \tag{C.5}$$

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JHEP11(2018)173

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